3.11 Nie-lineêre modelle

3.11 Non-linear models

Nie-lineêre vere, stywe & pap vere, nie-lineêre pendulum: LEES / *Non-linear springs, hard & soft springs, non-linear pendulum: READ* **(pp 185-188)**

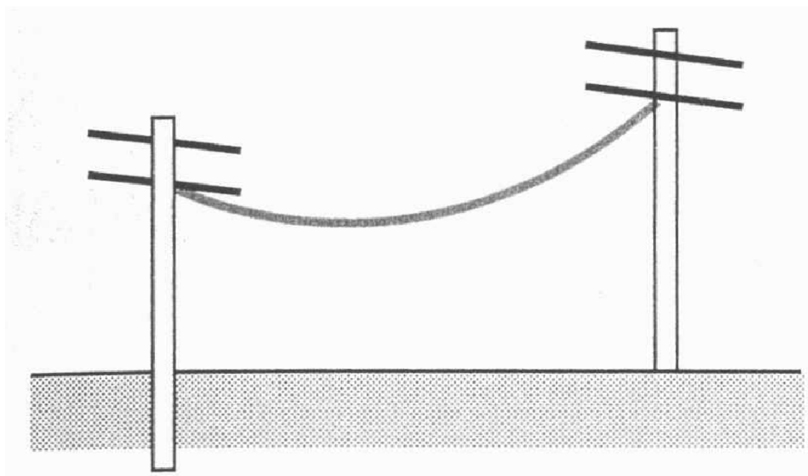
Kettinglyne

(bladsy 188)

Catenaries

(page 188)

Probleem: Voorspel die vorm van 'n swaar hangende buigbare kabel, bv. 'n telefoonlyn / *Problem: Predict the form of a heavy hanging flexible cable, e.g. a telephone cable*

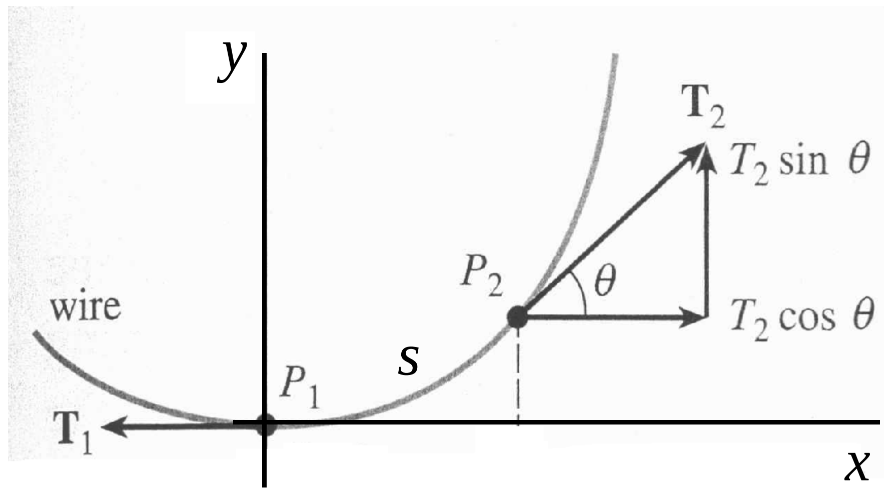


Aanname / *Assumption:*

Die kabel is uniform met 'n konstante gewig per eenheidslengte ρ

The cable is uniform with a constant **weight** *per unit length* ρ

Kies die oorsprong van die assestelsel by die laagste punt van die hangende kabel/draad / *Choose the origin of the system of axes at the lowest point of the hanging cable/wire*



(**Vir ewewig**/*For equilibrium*) $\boxed{(\rightarrow)\,\Sigma F_x = 0}$ $\quad -T_1 + T_2\cos\theta = 0$

(**Vir ewewig**/*For equilibrium*) $\boxed{(\uparrow)\,\Sigma F_y = 0}$ $\quad T_2\sin\theta - \rho s = 0$

Dus/*Therefore:*

$$T_2\cos\theta = T_1 \qquad (1)$$

$$T_2\sin\theta = \rho s \qquad (2)$$

$$(2)/(1): \quad \Rightarrow \quad \tan\theta = \frac{\rho s}{T_1}$$

$$\Rightarrow \quad \boxed{\frac{dy}{dx} = \frac{\rho s}{T_1}}$$

Neem eers aan dat die insakking baie klein is / *First assume that the sag is very small* $\Rightarrow s \approx x$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\rho}{T_1}x \quad \textbf{met}/\textit{with} \quad y(0) = 0 \quad \textbf{en}/\textit{and} \quad y'(0) = 0$$

Los op met skeiding van veranderlikes / *Solve with separation of variables*

$$\Rightarrow \; y(x) = \frac{\rho}{2T_1}x^2 \; \Rightarrow$$ **Paraboliese kettinglyn** / *Parabolic catenary*

Neem nou aan dat die insakking groot is / *Now assume that the sag is large*

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\rho}{T_1}s \quad \textbf{met}/\textit{with} \quad y(0) = 0 \quad \textbf{en}/\textit{and} \quad y'(0) = 0$$

Benodig dus uitdrukking vir booglengte $s = s(x)$ / *Therefore require expression for arc length* $s = s(x)$

$$\frac{d^2y}{dx^2} = \frac{\rho}{T_1}\frac{ds}{dx}$$

Benodig dus uitdrukking vir $\frac{ds}{dx}$ / *Therefore require expression for* $\frac{ds}{dx}$

$$\Delta s^2 = \Delta x^2 + \Delta y^2$$

$$\Rightarrow \frac{\Delta s}{\Delta x} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$$ **en neem/** *and take* $$\lim_{\Delta x \to 0} \Rightarrow \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\Rightarrow \boxed{\frac{d^2y}{dx^2} = \frac{\rho}{T_1}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}$$ (**nie-lineêr!/** *non-linear!*)

Aanvangsvoorwaardes / *Initial conditions:* $y(0) = 0$ **en/** *and* $y'(0) = 0$

Stel/ *Set* $u = \frac{dy}{dx} \quad \Rightarrow \quad \frac{du}{dx} = \frac{\rho}{T_1}\sqrt{1 + u^2}$

Los op met skeiding van veranderlikes / *Solve with separation of variables:*

$$\int \frac{du}{\sqrt{1 + u^2}} = \frac{\rho}{T_1}\int dx + C$$

$$\operatorname{arcsinh}(u) = \frac{\rho}{T_1}x + C$$ **(APP3, nr 17)**

As/ *If* $x = 0$, **dan/** *then* $y' = 0 \Rightarrow u = 0 \Rightarrow C = 0$

$$\Rightarrow \quad u = \sinh(\frac{\rho}{T_1}x)$$

$$\Rightarrow \quad \frac{dy}{dx} = \sinh\left(\frac{\rho}{T_1}x\right)$$

$$\Rightarrow \quad y = \frac{T_1}{\rho}\cosh\left(\frac{\rho}{T_1}x\right) + D$$ **(APP2, nr 20)**

As/*If* $x = 0$, **dan/***then* $y = 0 \Rightarrow D = -\frac{T_1}{\rho}$

$$\Rightarrow \quad \boxed{y(x) = \frac{T_1}{\rho}\left(\cosh\left(\frac{\rho}{T_1}x\right) - 1\right)}$$

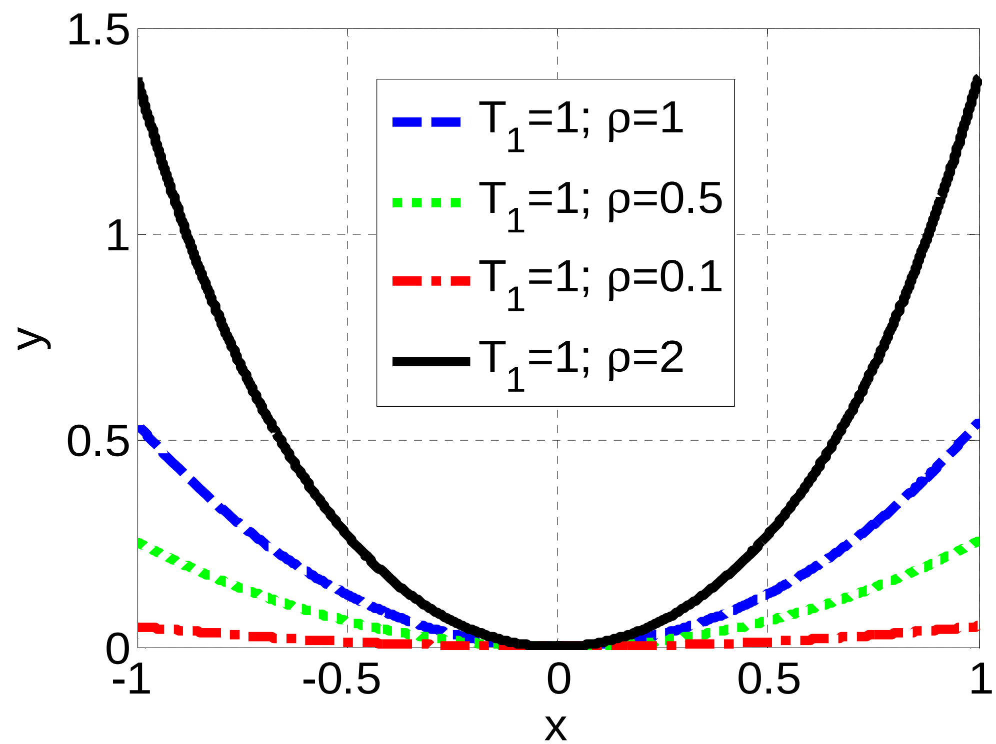
$\Rightarrow$ **Hiperboliese kettinglyn / ***Hiperbolic catenary*

Toets/*Test:* **As/***If* $\rho \ll T_1$ (**baie klein insakking /***very small sag*)...

Taylor-reeks/*Taylor series:* $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots$

$$\Rightarrow \quad y(x) = \frac{T_1}{\rho}\left(1 + \frac{1}{2}\left(\frac{\rho}{T_1}x\right)^2 - 1\right)$$
$$= \frac{\rho}{2T_1}x^2 \quad \textbf{(Parabolies/}\textit{Parabolic}\textbf{)}$$

Stem dus ooreen met resultaat vir $s \approx x$! / *This agrees with the result for $s \approx x$!*



Addisionele formules (nie in Z&W) / *Additional formulas (not in Z&W)*

Onthou/*Recall:*

$$T_2 \cos\theta = T_1 \quad (3)$$
$$T_2 \sin\theta = \rho s \quad (4)$$

$(3)^2 + (4)^2$**:** $\Rightarrow \quad {T_2}^2 = {T_1}^2 + \rho^2 s^2$

Onthou —— Formule vir die lengte van die kabel / *Recall —— Formula for the length of the cable:*

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

maar (soos bewys) / *but (as proved):*

$$\frac{dy}{dx} = \sinh\left(\frac{\rho}{T_1} x\right)$$

$$\frac{ds}{dx} = \sqrt{1 + \sinh^2\left(\frac{\rho}{T_1} x\right)}$$

maar/*but* $\cosh^2 x - \sinh^2 x = 1$, **dus/***therefore*

$$\frac{ds}{dx} = \cosh\left(\frac{\rho}{T_1}x\right)$$

$$s = \frac{T_1}{\rho}\sinh\left(\frac{\rho}{T_1}x\right) + C$$

As/*If* $x = 0$ **dan/***then* $s = 0 \quad \Rightarrow \quad C = 0$

$$\Rightarrow \quad s = \frac{T_1}{\rho}\sinh\left(\frac{\rho}{T_1}x\right)$$

(Alternatiewelik) Onthou / *(Alternatively) Recall:*

$$\frac{dy}{dx} = \frac{\rho}{T_1}s$$

$$\Rightarrow \quad s = \frac{T_1}{\rho}\frac{dy}{dx} = \frac{T_1}{\rho}\sinh\left(\frac{\rho}{T_1}x\right) \tag{5}$$

Formule vir die trekkrag in die kabel / *Formula for the tension in the cable:*

$$\begin{aligned} {T_2}^2 &= {T_1}^2 + \rho^2 \frac{T_1^2}{\rho^2} \sinh^2\left(\frac{\rho}{T_1}x\right) \\ &= {T_1}^2 \left(1 + \sinh^2\left(\frac{\rho}{T_1}x\right)\right) \\ &= {T_1}^2 \left(\cosh^2\left(\frac{\rho}{T_1}x\right)\right) \end{aligned}$$

$$\Rightarrow \quad T_2 = T_1 \cosh\left(\frac{\rho}{T_1}x\right)$$

maar/*but*

$$y(x) = \frac{T_1}{\rho}\left(\cosh\left(\frac{\rho}{T_1}x\right) - 1\right) \tag{6}$$

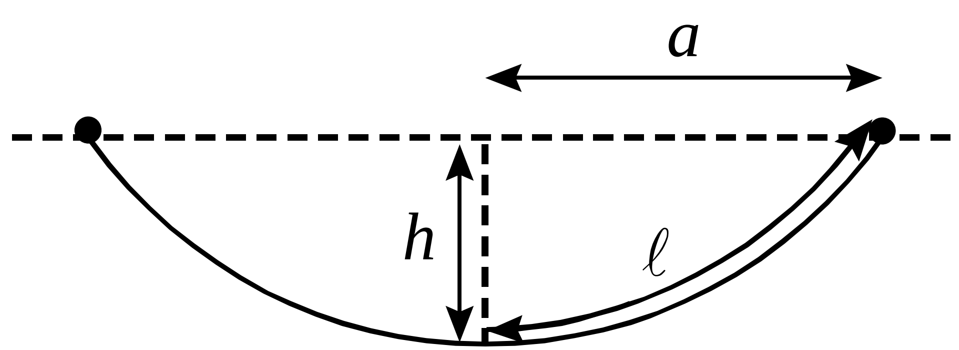
dus/*therefore*

$$T_2 = T_1\left(\frac{\rho}{T_1}y + 1\right)$$

dus/*therefore*

$$\boxed{T_2 = T_1 + \rho y}$$

Tipiese situasie: Simmetrie / *Typical situation: Symmetry*



Laat/*Let* $c = \dfrac{T_1}{\rho}$

$a \equiv$ **span;** $h \equiv$ **insakking/***sag***;** $\ell \equiv \frac{1}{2}$ **lengte van die kabel/***length of the cable*

Uit/*From* (6)**:**
$$h = c[\cosh(a/c) - 1]$$

Uit/*From* (5)**:**
$$\ell = c\sinh(a/c)$$

$\Rightarrow$ **2 vergelykings met 4 onbekendes; as 2 onbekendes gegee is, kan die ander 2 dus gevind word! /** *2 equations with 4 unknowns; if 2 unknowns are given, the other 2 can therefore be found!*

Probleem/*Problem* **1: Gegee/***Given* ℓ **&** h**, vind/***find* a **&** c

Probleem/*Problem* **2: Gegee/***Given* a **&** h**, vind/***find* ℓ **&** c

Probleem/*Problem* **3: Gegee/***Given* a **&** ℓ**, vind/***find* h **&** c