Problem 1: Given ℓ & h, find a & c

$$\boxed{h} = c[\cosh(a/c) - 1] \tag{1}$$

$$\boxed{\ell} = c\sinh(a/c) \tag{2}$$

From (1):

$$h + c = c\cosh(a/c) \tag{3}$$

$(3)^2 - (2)^2$:

$$c = \frac{\ell^2 - h^2}{2h}$$

Substitute back into (2):

$$a = c\,\mathrm{arcsinh}(\ell/c)$$

Example: Calculate a and c when the length of the deflected cable equals 3.3 m and the sag equals 0.6 m.

Answer: $a = 1.5$ m; $c = 1.9688$ m

Problem 2: Given a & h, find ℓ & c

$$\boxed{h} = c[\cosh(\boxed{a}/c) - 1] \tag{4}$$

$$\ell = c\sinh(\boxed{a}/c) \tag{5}$$

Equation (4): One equation, one unknown

Sol options: (1) Numerical (more accurate)
(2) Formula for small sag
(approximation: $\cosh x \approx 1 + \frac{x^2}{2!}$)

Example: The span and sag of a flexible cable are 150 m and 60 m, respectively. The cable has a mass per unit length of 12 kg/m. Determine
(i) the tension T_1 at the lowest point of the deflected cable,
(ii) the maximum tension in the deflected cable, and
(iii) the length of the deflected cable.

$$60 = c\,(\cosh(150/c) - 1)$$

Sol option 2: Assume that the sag is small

($c = \dfrac{T_1}{\rho}$ large)

$$\Rightarrow \quad 60 = c\left(1 + \frac{1}{2}(150/c)^2 - 1\right)$$

$$\Rightarrow \quad c = 187.5\,\text{m} \quad \text{(correct value: } 196.8\,\text{m)}$$

$$T_1 = c\rho = 187.5 \times 12 \times 9.81 = 22\,073\,\text{N}$$

(correct value: $23\,167\,\text{N}$)

$$T_2 = T_1 + \rho h = 22\,073 + (12)(9.81)(60) = 29\,136\,\text{N}$$

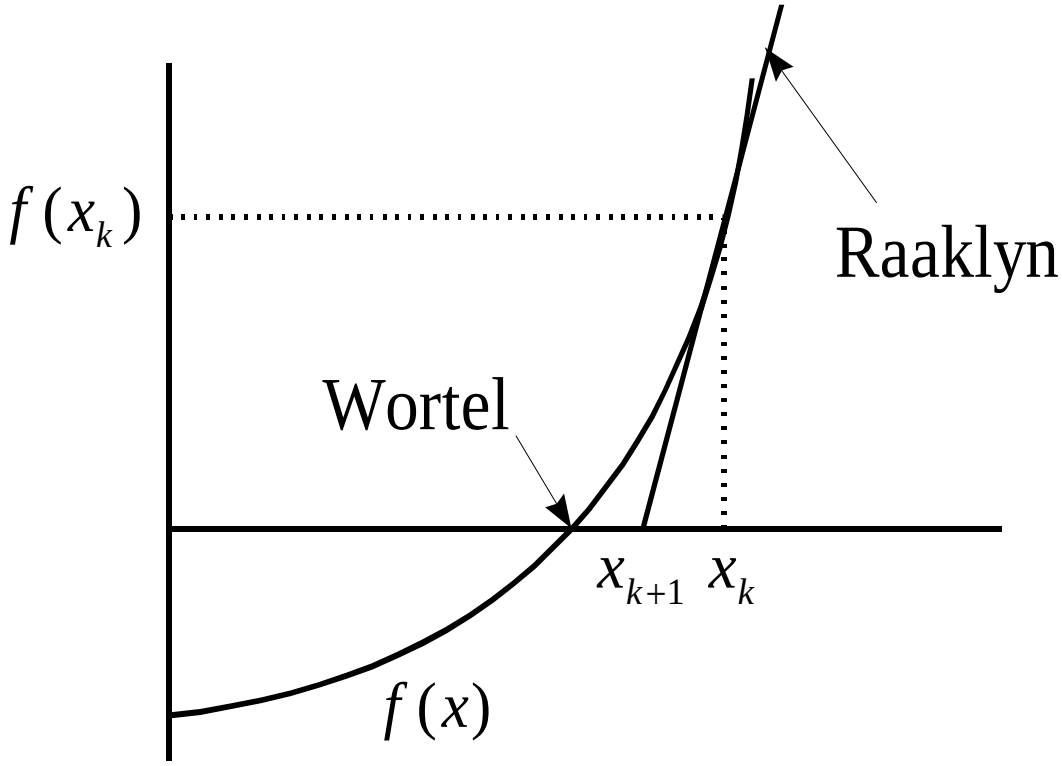
(correct value: $30\,230\,\text{N}$)

$$\ell = 187.5\,\sinh(150/187.5) = 166.5\,\text{m}$$

The length is therefore $333\,\text{m}$ (correct value: $330\,\text{m}$)

Sol option 1: Numerical (more accurate)

Newton's method: $f(x) = 0$



$$f'(x_k) = \frac{f(x_k) - 0}{x_k - x_{k+1}}$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$x_0, \quad x_1, \quad x_2, \quad x_3 \quad \rightarrow \quad \text{root}$$

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$$60 = c\,(\cosh(150/c) - 1)$$

$$f(c) = c\,(\cosh(150/c) - 1) - 60 = 0$$

$$f'(c) = (\cosh(150/c) - 1) - \frac{150}{c}(\sinh(150/c))$$

Choose $c_0 = 187.5$ (approximation for small sag)

$$c_1 = c_0 - \frac{f(c_0)}{f'(c_0)} = 196.3$$

$$c_2 = 196.8; \quad c_3 = 196.8$$

$$\Rightarrow c = 196.8\,\text{m}$$

Matlab output:

Iteration	c	$f(c)$	$f'(c)$
0	187.5000	3.2691	-0.3730
1	196.2630	0.1661	-0.3361
2	196.7574	0.0005	-0.3342
3	196.7588	0.0000	-0.3342

Problem 3: Given a & ℓ, find h & c

$$h = c[\cosh(\boxed{a}/c) - 1] \tag{6}$$

$$\boxed{\ell} = c\sinh(\boxed{a}/c) \tag{7}$$

Example: Given $a = 5\,\mathsf{m}$ and $\ell = 20\,\mathsf{m}$. Find h and c.

$$20 = c\,\sinh(5/c)$$

Approximation for small sag: $\sinh(x) \approx x + \dfrac{x^3}{3!}$

$$20 = c\left(\frac{5}{c} + \frac{1}{6}\left(\frac{5}{c}\right)^3\right)$$

$$\Rightarrow c = 1.1785\,\mathsf{m}$$

$$h = 1.1785(\cosh(5/1.1785) - 1) \approx 39\,\mathsf{m}$$

Rather try Newton's method...

$$20 = c\,\sinh(5/c)$$

$$f(c) = c\,\sinh(5/c) - 20 = 0$$

$$f'(c) = \sinh(5/c) - (5/c)\cosh(5/c)$$

Choose $c_0 = 1.1785$ (approximation for small sag)

$$c_1 = c_0 - \frac{f(c_0)}{f'(c_0)} = 1.3645$$

Repeat... $\Rightarrow c = 1.5320\,\text{m}$

$$h = 1.5320\,(\cosh(5/1.5320) - 1) = 18.53\,\text{m}$$

Matlab output:

Iteration	c	$f(c)$	$f'(c)$
0	1.1785	20.999	-112.873
1	1.3645	6.6086	-52.0460
⋮	⋮	⋮	⋮
5	1.5320	0.0000	-29.6791