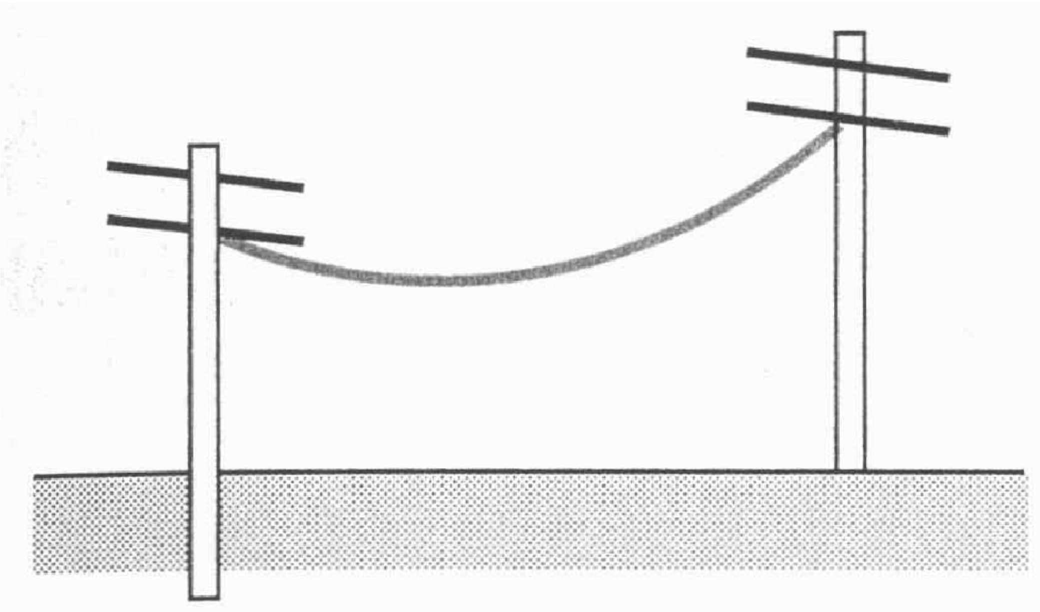
3.11 Non-linear models

Non-linear springs, hard and soft springs, non-linear pendulum (pp. 185-188): READ

<u>Catenaries</u> *(page 188)*

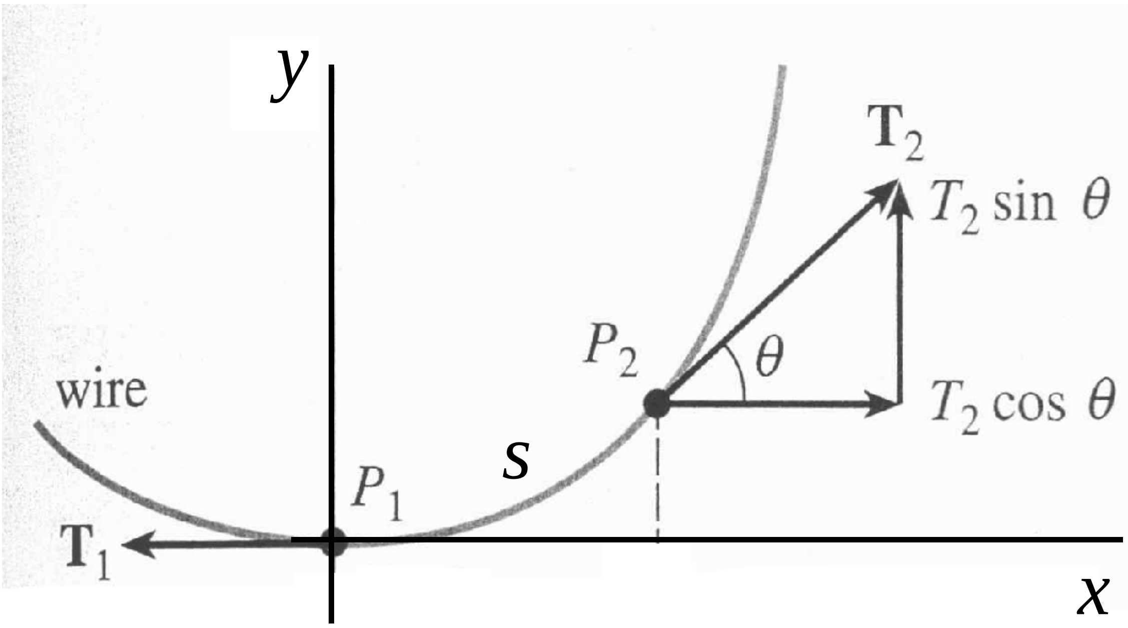
Problem: Predict the form of a heavy hanging flexible cable, e.g. a telephone cable



Assumption:

The cable is uniform with a constant **weight** per unit length ρ

Choose the origin of the system of axes at the lowest point of the hanging cable/wire



(For eqm) $\boxed{(\rightarrow)\,\Sigma F_x = 0}$ $\quad -T_1 + T_2\cos\theta = 0$

(For eqm) $\boxed{(\uparrow)\,\Sigma F_y = 0}$ $\quad T_2\sin\theta - \rho s = 0$

Thus:

$$T_2\cos\theta = T_1 \qquad (1)$$
$$T_2\sin\theta = \rho s \qquad (2)$$

$$(2)/(1): \quad \Rightarrow \quad \tan\theta = \frac{\rho s}{T_1}$$

$$\Rightarrow \quad \boxed{\frac{dy}{dx} = \frac{\rho s}{T_1}}$$

First assume that the sag is **very small**

$\Rightarrow s \approx x$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\rho}{T_1} x \ \text{ with } \ y(0) = 0 \ \text{ and } \ y'(0) = 0$$

Solve with separation of variables

$$\Rightarrow \ y(x) = \frac{\rho}{2T_1} x^2 \ \Rightarrow \ \textbf{Parabolic catenary}$$

Now assume that the sag is **large**

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\rho}{T_1} s \ \text{ with } \ y(0) = 0 \ \text{ and } \ y'(0) = 0$$

Therefore require expression for arc length $s = s(x)$

$$\frac{d^2y}{dx^2} = \frac{\rho}{T_1} \frac{ds}{dx}$$

Therefore require expression for $\dfrac{ds}{dx}$

$$\Delta s^2 = \Delta x^2 + \Delta y^2$$

$$\Rightarrow \quad \frac{\Delta s}{\Delta x} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \quad \text{and take} \quad \lim_{\Delta x \to 0}$$

$$\Rightarrow \quad \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\Rightarrow \quad \boxed{\frac{d^2y}{dx^2} = \frac{\rho}{T_1}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \quad \text{(non-linear!)}$$

Initial conditions: $y(0) = 0$ and $y'(0) = 0$

Set $u = \dfrac{dy}{dx} \quad \Rightarrow \quad \dfrac{du}{dx} = \dfrac{\rho}{T_1}\sqrt{1 + u^2}$

Solve with separation of variables:

$$\int \frac{du}{\sqrt{1 + u^2}} = \frac{\rho}{T_1} \int dx + C$$

$$\operatorname{arcsinh}(u) = \frac{\rho}{T_1} x + C \quad \text{(APP3, nr 17)}$$

If $x = 0$, then $y' = 0 \Rightarrow u = 0 \Rightarrow C = 0$

$$\Rightarrow \quad u = \sinh(\frac{\rho}{T_1}x)$$

$$\Rightarrow \quad \frac{dy}{dx} = \sinh\left(\frac{\rho}{T_1}x\right)$$

$$\Rightarrow \quad y = \frac{T_1}{\rho}\cosh\left(\frac{\rho}{T_1}x\right) + D$$ (APP2, nr 20)

If $x = 0$, then $y = 0 \Rightarrow D = -\frac{T_1}{\rho}$

$$\Rightarrow \quad \boxed{y(x) = \frac{T_1}{\rho}\left(\cosh\left(\frac{\rho}{T_1}x\right) - 1\right)}$$

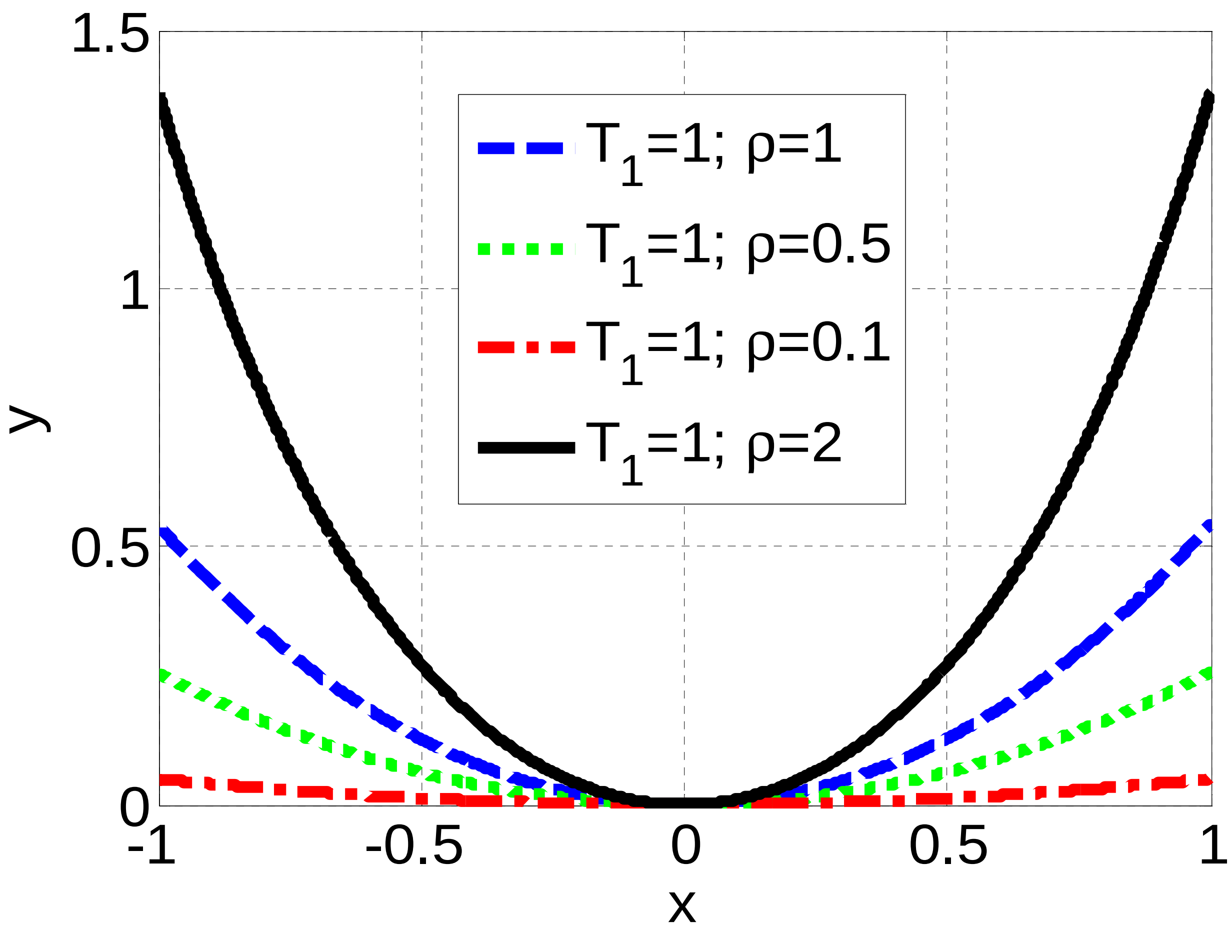
$\Rightarrow$ **Hiperbolic catenary**

Test: If $\rho \ll T_1$ (very small sag)...

Taylor series: $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots$

$$\Rightarrow \quad y(x) = \frac{T_1}{\rho}\left(1 + \frac{1}{2}\left(\frac{\rho}{T_1}x\right)^2 - 1\right)$$
$$= \frac{\rho}{2T_1}x^2 \quad \text{(Parabolic)}$$

This agrees with the result for $s \approx x$!



Additional formulas (not in Z&W)

Recall:

$$T_2 \cos\theta = T_1 \tag{3}$$
$$T_2 \sin\theta = \rho s \tag{4}$$

$(3)^2 + (4)^2$: $\quad \Rightarrow \quad {T_2}^2 = {T_1}^2 + \rho^2 s^2$

Recall $--$ Formula for the length of the cable:

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

but (as proved):

$$\frac{dy}{dx} = \sinh\left(\frac{\rho}{T_1}x\right)$$

$$\frac{ds}{dx} = \sqrt{1 + \sinh^2\left(\frac{\rho}{T_1}x\right)}$$

but $\cosh^2 x - \sinh^2 x = 1$, therefore

$$\frac{ds}{dx} = \cosh\left(\frac{\rho}{T_1}x\right)$$

$$s = \frac{T_1}{\rho}\sinh\left(\frac{\rho}{T_1}x\right) + C$$

If $x = 0$ then $s = 0 \quad \Rightarrow \quad C = 0$

$$\Rightarrow \quad s = \frac{T_1}{\rho}\sinh\left(\frac{\rho}{T_1}x\right)$$

(Alternatively) Recall:

$$\frac{dy}{dx} = \frac{\rho}{T_1}s$$

$$\Rightarrow \quad s = \frac{T_1}{\rho}\frac{dy}{dx} = \frac{T_1}{\rho}\sinh\left(\frac{\rho}{T_1}x\right) \tag{5}$$

Formula for the tension in the cable:

$$
\begin{aligned}
{T_2}^2 &= {T_1}^2 + \rho^2 \frac{T_1^2}{\rho^2} \sinh^2\left(\frac{\rho}{T_1}x\right) \\
&= {T_1}^2 \left(1 + \sinh^2\left(\frac{\rho}{T_1}x\right)\right) \\
&= {T_1}^2 \left(\cosh^2\left(\frac{\rho}{T_1}x\right)\right)
\end{aligned}
$$

$$\Rightarrow \quad T_2 = T_1 \cosh\left(\frac{\rho}{T_1}x\right)$$

but

$$y(x) = \frac{T_1}{\rho}\left(\cosh\left(\frac{\rho}{T_1}x\right) - 1\right) \tag{6}$$

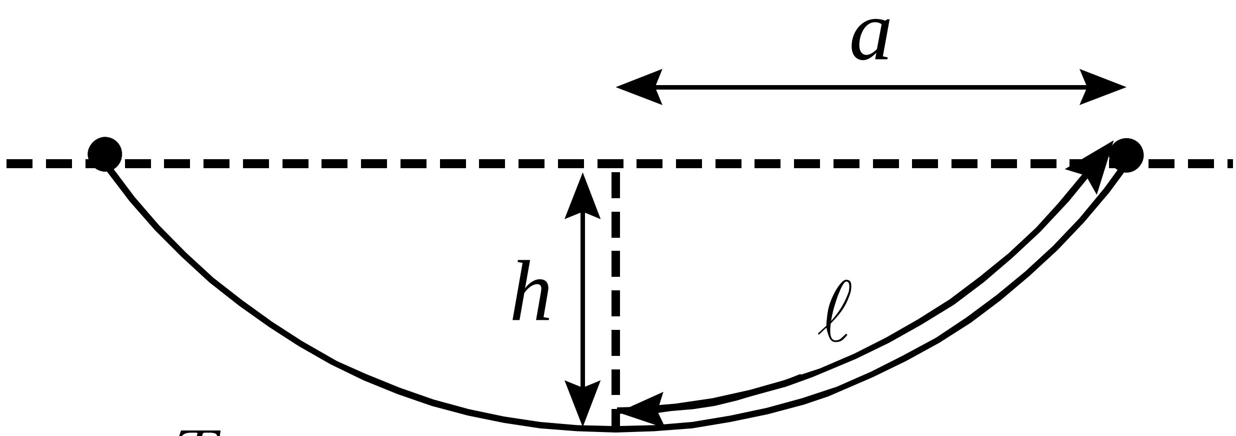
therefore

$$T_2 = T_1\left(\frac{\rho}{T_1}y + 1\right)$$

therefore

$$\boxed{T_2 = T_1 + \rho y}$$

Typical situation: Symmetry



Let $c = \dfrac{T_1}{\rho}$

$a \equiv$ span; $h \equiv$ sag; $\ell \equiv \frac{1}{2}$ length of the cable

From (6): $$h = c[\cosh(a/c) - 1]$$

From (5): $$\ell = c\sinh(a/c)$$

$\Rightarrow$ 2 equations with 4 unknowns; if 2 unknowns are given, the other 2 can therefore be found!

<u>Problem 1:</u> Given ℓ & h, find a & c
<u>Problem 2:</u> Given a & h, find ℓ & c
<u>Problem 3:</u> Given a & ℓ, find h & c