THE CONTINUOUS FOURIER TRANSFORM

In 1 dimension:

$$\textbf{FT}\,\{f(x)\} = F(u) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i u x}\, dx$$

$$\textbf{IFT}\,\{F(u)\} = f(x) = \int_{-\infty}^{\infty} F(u)\, e^{2\pi i u x}\, du$$

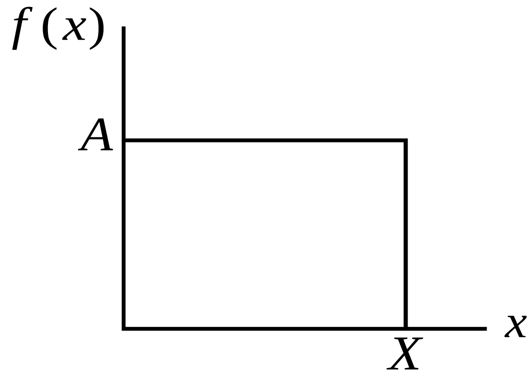
In 2 dimensions:

$$\textbf{FT}\,\{f(x,y)\} = F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-2\pi i (ux+vy)}\, dx\, dy$$

$$\textbf{IFT}\,\{F(u,v)\} = f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\, e^{2\pi i (ux+vy)}\, du\, dv$$

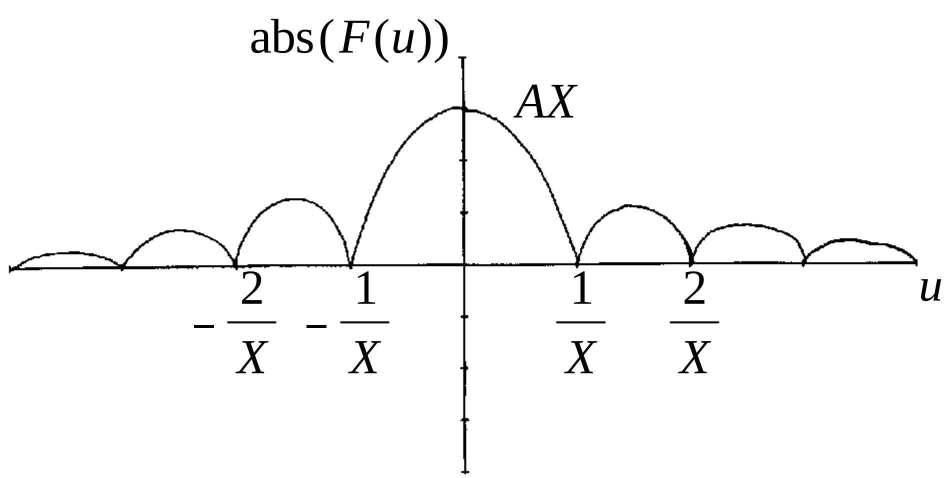
Example 4

Calculate and sketch the Fourier spectrum of $f(x) = \begin{cases} A, & \textbf{if } x \in [0, X] \\ 0, & \textbf{otherwise} \end{cases}$



$$\begin{aligned} F(u) &= \int_{-\infty}^{\infty} f(x) e^{-2\pi i u x} dx \\ &= \vdots \\ &= \frac{A}{\pi u} \sin(\pi u X) e^{-\pi i u X} \end{aligned}$$

$$\begin{aligned} |F(u)| &= \left| \frac{A}{\pi u} \right| |\sin(\pi u X)| \overbrace{\left| e^{-\pi i u X} \right|}^{=1} \\ &= AX \left| \frac{\sin(\pi u X)}{\pi u X} \right| \end{aligned}$$



What happens to $f(x)$ and $|F(u)|$ for small and large values of X? Explain this phenomenon

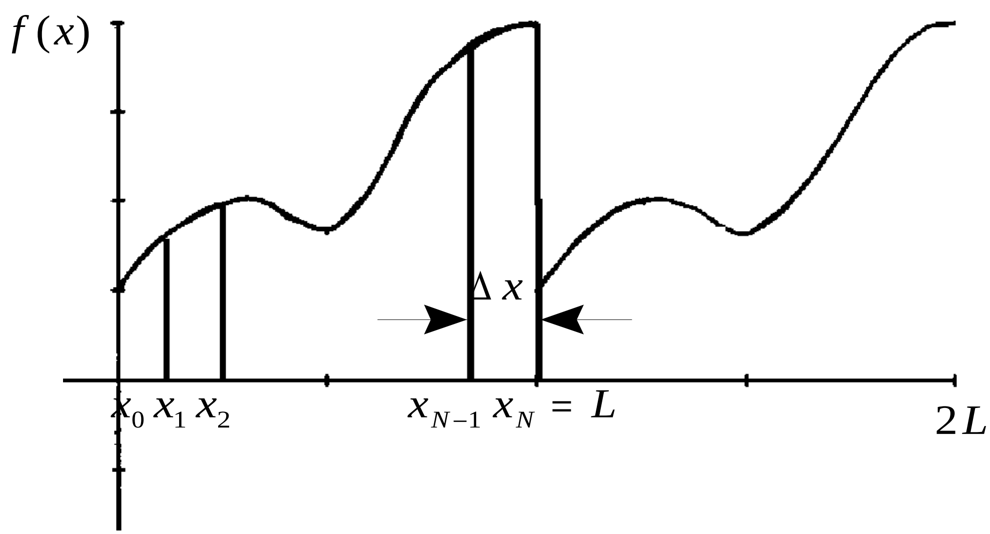
THE DISCRETE FOURIER TRANSFORM

On a computer we have two restrictions:

(1) All functions are discrete, that is VECTORS

(2) These functions are defined on a FINITE interval

We again consider a continuous function, $f(x)$, this time defined on the interval $[0, L]$, and proceed to discretise this function, defining it only at certain points x_n that are Δx units apart



The Fourier series for $f(x)$ **is** $\tilde{f}(x) = \sum_{n=-\infty}^{\infty} d_n \, e^{2\pi i n x/L}$

Note that

$$\int_0^L e^{2\pi i n x/L} e^{-2\pi i k x/L} dx = \begin{cases} 0, & \textbf{if } n \neq k \\ L, & \textbf{if } n = k \end{cases}$$

And as before

$$\begin{aligned} d_n &= \frac{1}{L}\int_0^L f(x) e^{-2\pi i n x/L} dx \\ &\approx \frac{\Delta x}{L}\left(\frac{1}{2} f(x_0) e^{-2\pi i n x_0/L} + f(x_1) e^{-2\pi i n x_1/L} + \ldots + \frac{1}{2} f(x_N) e^{-2\pi i n x_N/L}\right) \end{aligned}$$

(Trapezium rule for numerical integration)

$$\approx \frac{\Delta x}{L}\sum_{j=0}^{N-1} f(x_j) e^{-2\pi i n x_j/L} \quad \textbf{(Since } f(x_{j+N}) = f(x_j) \ \forall \ j\textbf{)}$$

$$\Delta x = \frac{L}{N} \quad \Rightarrow \quad \frac{\Delta x}{L} = \frac{1}{N} \qquad\qquad x_j = j\Delta x = \frac{jL}{N} \quad \Rightarrow \quad \frac{x_j}{L} = \frac{j}{N}$$

Let $f(x_j) = f(j) = f_j$**, then** $d_n = \frac{1}{N}\sum_{j=0}^{N-1} f_j e^{-2\pi i n j/N} = F_n$

We therefore define the one dimensional DFT as follows:

$$F_n = \frac{1}{N}\sum_{j=0}^{N-1} f_j e^{-2\pi i n j/N}, \quad n = 0, \ldots, N-1$$

OR

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\, e^{-2\pi i u x/N}, \quad u = 0, \ldots, N-1$$

Derivation of the DIFT: **Multiply both sides with** $e^{2\pi i n k/N}$ **and calculate** $\sum_{n=0}^{N-1}$**:**

$$\begin{aligned}\sum_{n=0}^{N-1} F_n\, e^{2\pi i n k/N} &= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{j=0}^{N-1} f_j\, e^{-2\pi i n j/N}\, e^{2\pi i n k/N} \\ &= \frac{1}{N}\sum_{j=0}^{N-1} f_j \sum_{n=0}^{N-1} e^{2\pi i n(k-j)/N}\end{aligned}$$

Let $r = e^{2\pi i(k-j)/N}$**, so** $\sum_{n=0}^{N-1} e^{2\pi i n(k-j)/N} = \sum_{n=0}^{N-1} r^n = \begin{cases} \dfrac{r^N - 1}{r-1}, & \textbf{if } r \neq 1 \\ N, & \textbf{if } r = 1 \end{cases}$

Note that **If** $r = 1$, **then** $j = k$
If $r \neq 1$, **then** $j \neq k$

So

$$\sum_{n=0}^{N-1} e^{2\pi i n(k-j)/N} = \begin{cases} N, & \textbf{if } j = k \\ \dfrac{e^{2\pi i(k-j)} - 1}{e^{2\pi i(k-j)/N} - 1}, & \textbf{if } j \neq k \end{cases}$$

Note that $e^{2\pi i(k-j)} = 1$**, therefore**

$$\sum_{n=0}^{N-1} e^{2\pi i n(k-j)/N} = \begin{cases} N, & \textbf{if } j = k \\ 0, & \textbf{if } j \neq k \end{cases}$$

$$\Rightarrow \quad \sum_{n=0}^{N-1} F_n \, e^{2\pi i n k/N} = \frac{1}{N}(f_k)N$$

$$\Rightarrow \quad f_k = \sum_{n=0}^{N-1} F_n \, e^{2\pi i n k/N}$$

DFT: $$F_n = \frac{1}{N}\sum_{j=0}^{N-1} f_j \, e^{-2\pi i n j/N}, \quad n = 0, \ldots, N-1$$

DIFT: $$f_j = \sum_{n=0}^{N-1} F_n \, e^{2\pi i n j/N}, \quad j = 0, \ldots, N-1$$

Note that $$f_{j+N} = f_j \quad \forall\, j \in \mathbb{Z}$$ $$F_{n+N} = F_n \quad \forall\, n \in \mathbb{Z}$$ **(verify)**

Similarly for the two dimensional case (here the functions are matrices, images), we have:

DFT: $$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\, e^{-2\pi i(ux/M + vy/N)}$$

DIFT: $$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)\, e^{2\pi i(ux/M + vy/N)}$$

M **represents the number of rows and** N **the number of columns**

Again note that
$$f(x+M, y+N) = f(x,y) \quad \forall\, x, y \in \mathbb{Z}$$
$$F(u+M, v+N) = F(u,v) \quad \forall\, u, v \in \mathbb{Z}$$

Matrix notation for the one dimensional case

$$\begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_{N-1} \end{bmatrix} = \frac{1}{N} \begin{bmatrix} w^0 & w^0 & w^0 & w^0 & \dots & w^0 \\ w^0 & w^1 & w^2 & w^3 & \dots & w^{N-1} \\ w^0 & w^2 & w^4 & w^6 & \dots & w^{2(N-1)} \\ w^0 & w^3 & w^6 & w^9 & \dots & w^{3(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ w^0 & w^{N-1} & w^{2(N-1)} & w^{3(N-1)} & \dots & w^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{N-1} \end{bmatrix}$$

where $w = e^{-2\pi i/N}$**, or alternatively:**

DFT : $\mathbf{F} = \frac{1}{N}\Phi\mathbf{f}$

DIFT : $\mathbf{f} = \overline{\Phi}\mathbf{F}$

Example 5: Calculate the DFT of $[2 \quad 3 \quad 4 \quad 4]^T$

$$\begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} w^0 & w^0 & w^0 & w^0 \\ w^0 & w^1 & w^2 & w^3 \\ w^0 & w^2 & w^4 & w^6 \\ w^0 & w^3 & w^6 & w^9 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \\ 4 \end{bmatrix}$$

$$\mathbf{F} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 3.25 \\ -0.5+0.25i \\ -0.25 \\ -0.5-0.25i \end{bmatrix} \quad \Rightarrow \quad |\mathbf{F}| = \begin{bmatrix} 3.25 \\ \sqrt{0.375} \\ 0.25 \\ \sqrt{0.375} \end{bmatrix}$$

PROPERTIES OF THE 2-D DFT

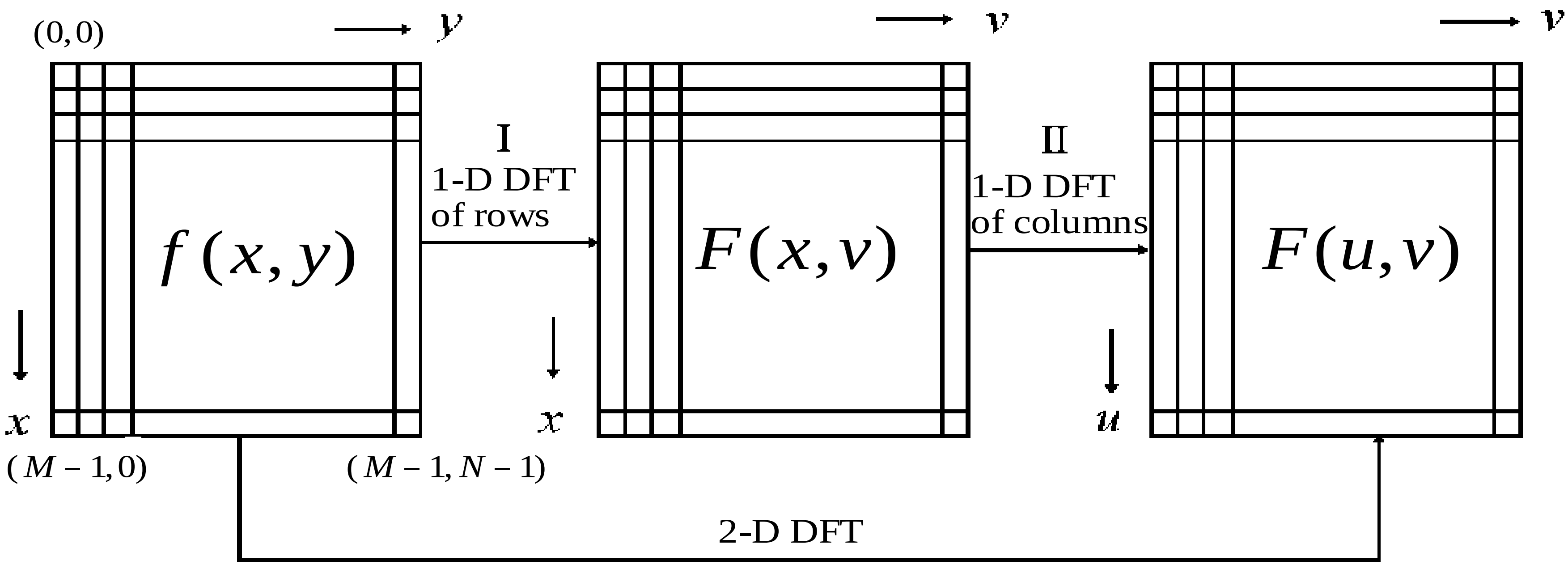
(G&W: page 258, section 4.6)

(1) Separability

$$\begin{aligned} F(u,v) &= \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\, e^{-2\pi i(ux/M+vy/N)} \\ &= \frac{1}{M}\sum_{x=0}^{M-1} e^{-2\pi iux/M}\, \frac{1}{N}\sum_{y=0}^{N-1} f(x,y)\, e^{-2\pi ivy/N} \\ &= \frac{1}{M}\sum_{x=0}^{M-1} e^{-2\pi iux/M}\, F(x,v) \end{aligned}$$

$$\boxed{\mathbf{I}} \quad F(x,v) = \frac{1}{N}\sum_{y=0}^{N-1} f(x,y)\, e^{-2\pi ivy/N} \quad , \; x = 0,\ldots,M-1$$

$$\boxed{\mathbf{II}} \quad F(u,v) = \frac{1}{M}\sum_{x=0}^{M-1} F(x,v)\, e^{-2\pi iux/M} \quad , \; v = 0,\ldots,N-1$$


(0,0)
y
v
v
I
1-D DFT
of rows
II
1-D DFT
of columns
f(x,y)
F(x,v)
F(u,v)
x
x
u
(M − 1,0)
(M − 1, N − 1)
2-D DFT

(2) Translation

$$
\begin{aligned}
&\textbf{FT}\left\{f(x,y)\ e^{2\pi i(u_0x/M+v_0y/N)}\right\}\\
&= \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\left\{f(x,y)\ e^{2\pi i(u_0x/M+v_0y/N)}\right\} e^{-2\pi i(ux/M+vy/N)}\\
&= \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\ e^{-2\pi i[(u-u_0)x/M+(v-v_0)y/N]}\\
&= F(u-u_0, v-v_0)
\end{aligned}
$$

Therefore

$$f(x,y)\ e^{2\pi i(u_0x/M+v_0y/N)} \Leftrightarrow F(u-u_0, v-v_0)$$

Similarly

$$f(x-x_0, y-y_0) \Leftrightarrow F(u,v)\ e^{-2\pi i(ux_0/M+vy_0/N)}$$

(verify)

Note that, if $u_0 = \dfrac{M}{2}$ and $v_0 = \dfrac{N}{2}$ then

$$f(x,y)(-1)^{x+y} \Leftrightarrow F\left(u - \frac{M}{2}, v - \frac{N}{2}\right)$$

This is exactly what `fftshift` does in MATLAB

Note that $|\textbf{FT}\{f(x - x_0,\, y - y_0)\}| = |F(u,v)|$ **(verify)**

Why is this significant?

(3) Periodicity/Symmetry

Periodicity

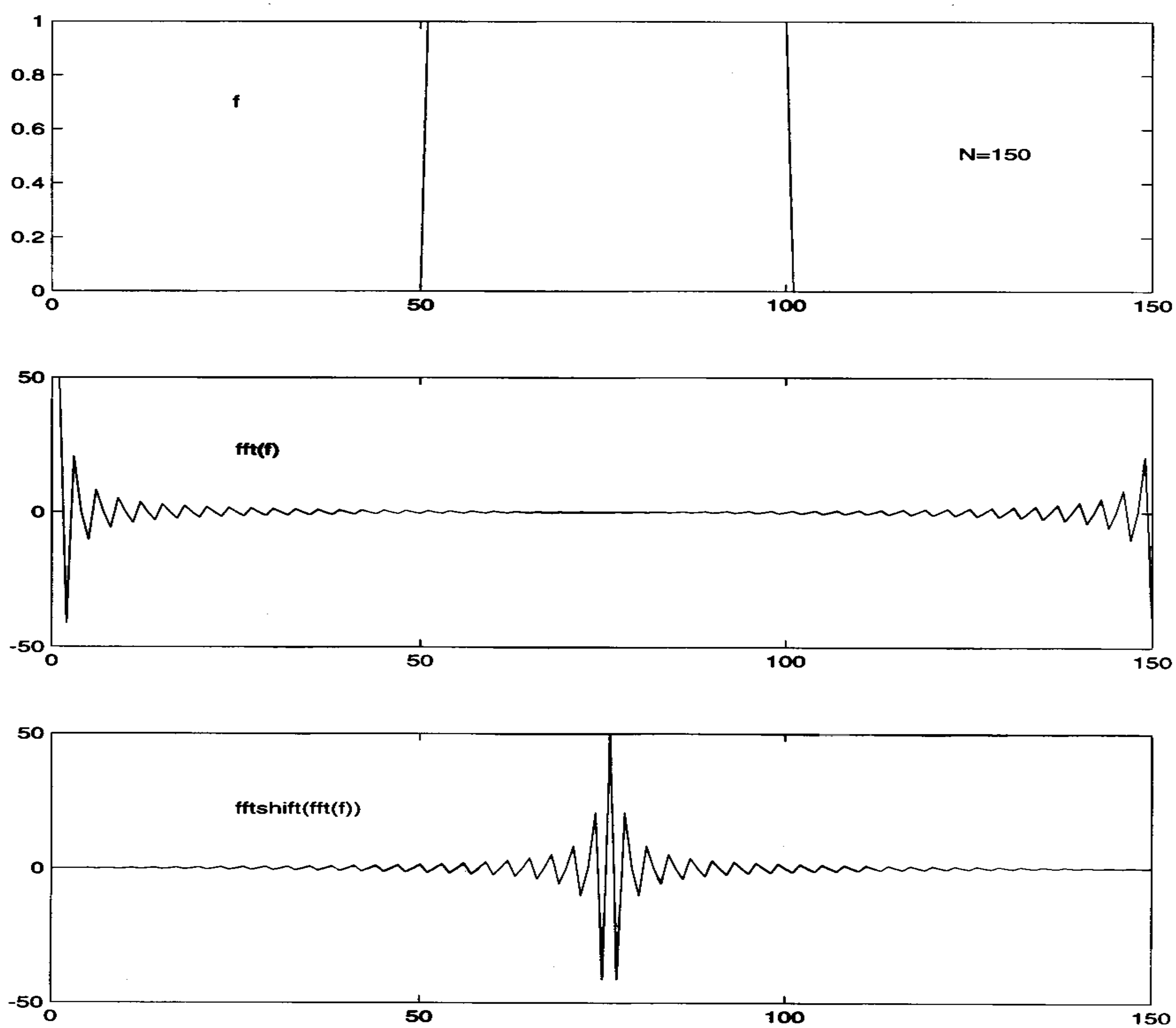
Remember that we have the following in two dimensions:

$$f(x+M, y+N) = f(x,y)$$

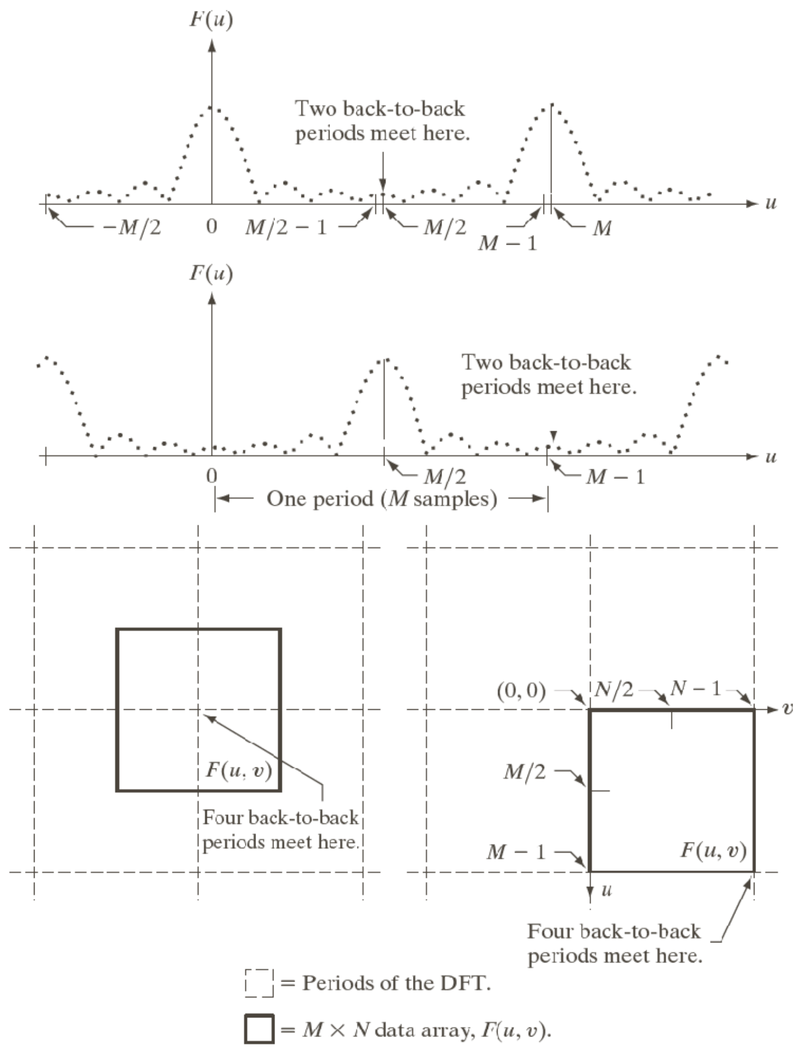
$$F(u+M, v+N) = F(u,v)$$

This implies that we only need to know $f(x,y)$ or $F(u,v)$ for one period

Symmetry: **In 1-D:** $|F(u)| = |F(-u)|$; **In 2-D:** $|F(u,v)| = |F(-u,-v)|$



Also see Figure 4.23 on p 260 of G&W



Remember: always display `abs( fftshift( FT ) )`**, but manipulate** `FT` **or** `fftshift( FT )`

(4) Rotation: **In polar coordinates we have** $f(r,\theta+\theta_0) \Leftrightarrow F(\omega,\phi+\theta_0)$

We prove this result for the continuous case only and first derive an expression for the Fourier transform in polar coordinates:

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-2\pi i(ux+vy)}dxdy$$

Let $x = r\cos\theta$**,** $y = r\sin\theta$**,** $u = \omega\cos\phi$**, and** $v = \omega\sin\phi$**, then**

$$\begin{aligned} F(\omega,\phi) &= \int_0^{2\pi}\int_0^{\infty} f(r,\theta)\, e^{-2\pi i(r\omega\cos\theta\cos\phi + r\omega\sin\theta\sin\phi)} r dr d\theta \\ &= \int_0^{2\pi}\int_0^{\infty} f(r,\theta)\, e^{-2\pi i r\omega\cos(\theta-\phi)} r dr d\theta \end{aligned}$$

Therefore

$$\textbf{FT}\left\{f(r,\theta+\theta_0)\right\} = \int_0^{2\pi}\int_0^{\infty} f(r,\theta+\theta_0)\, e^{-2\pi i r\omega\cos(\theta-\phi)} r dr d\theta$$

Let $\theta+\theta_0 = \overline{\theta}$**, then**

$$\begin{aligned} \textbf{FT}\left\{f(r,\theta+\theta_0)\right\} &= \int_0^{2\pi}\int_0^{\infty} f(r,\overline{\theta})\, e^{-2\pi i r\omega\cos\left(\overline{\theta}-(\theta_0+\phi)\right)} r dr d\overline{\theta} \\ &= F(\omega,\phi+\theta_0) \end{aligned}$$

(5) Distributivity and scaling

Distributivity

$$\mathbf{FT}\left\{a\, f_1(x,y) + b\, f_2(x,y)\right\} = a\,\mathbf{FT}\left\{f_1(x,y)\right\} + b\,\mathbf{FT}\left\{f_2(x,y)\right\}$$

Scaling

$$f(ax, by) \Leftrightarrow \frac{1}{|a\, b|}\, F\left(\frac{u}{a}, \frac{v}{b}\right)$$

We again prove this for the continuous case only

$$\mathbf{FT}\left\{f(ax,by)\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(ax,by)\, e^{-2\pi i(ux+vy)}\, dx\, dy$$

Let $g = ax$ **and** $h = by \Rightarrow dx = \frac{1}{a}\,dg$ **and** $dy = \frac{1}{b}\,dh$**, then**

$$\begin{aligned}\mathbf{FT}\left\{f(ax,by)\right\} &= \frac{1}{|a\, b|}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(g,h)\, e^{-2\pi i\left(\frac{u}{a}g+\frac{v}{b}h\right)}\, dg\, dh \\ &= \frac{1}{|a\, b|}\, F\left(\frac{u}{a}, \frac{v}{b}\right)\end{aligned}$$