Lemma

If a graph G of order $n \geq 3$ has a Hamiltonian $u-v$ path and $\deg(u) + \deg(v) \geq n$, then G is Hamiltonian.

Proof: Let $P = (u = x_1, x_2, \ldots, x_n = v)$ be a Hamiltonian $u-v$ path of G. Suppose that, for each x_i that is adjacent to u, x_{i-1} is not adjacent to v. Then $\deg(v) \leq n-1-\deg(u)$, hence $\deg(u)+\deg(v) \leq n-1$. Therefore, if $\deg(u) + \deg(v) \geq n$, then there exists an x_i adjacent to u such that x_{i-1} is adjacent to v, in which case $C = (x_1, x_i, x_{i+1}, \ldots, x_n, x_{i-1}, x_{i-2}, \ldots, x_1)$ is a Hamiltonian cycle of G ∎

Theorem 6.6

Let G be a graph of order $n \geq 3$.
If $\deg(u) + \deg(v) \geq n$ for every pair u, v of non-adjacent vertices of G, then G is Hamiltonian.

<u>Proof:</u> Assume to the contrary that G is a non-Hamiltonian graph of order $n \geq 3$ such that $\deg_G(u) + \deg_G(v) \geq n$ for every pair u, v of non-adjacent vertices of G. Let H be a non-Hamiltonian graph with a maximum number of edges that contains G as a subgraph. Let x and y be two non-adjacent vertices of H. Then $H + xy$ is Hamiltonian, and therefore H has a Hamiltonian x-y path. Furthermore, $\deg_H(x) + \deg_H(y) \geq \deg_G(x) + \deg_G(y) \geq n$. It follows from the Lemma that H is Hamiltonian, contradicting H non-Hamiltonian. ∎

<u>Theorem 6.8</u>

Let u and v be non-adjacent vertices in a graph G of order n such that $\deg_G(u) + \deg_G(v) \geq n$. Then $G + uv$ is Hamiltonian iff and only if G is Hamiltonian.

<u>Proof</u>: If G is Hamiltonian, then clearly $G+uv$ is also Hamiltonian. Now suppose $G+uv$ is Hamiltonian and let C be a Hamiltonian cycle of $G+uv$. If uv does not lie on C, then C is a Hamiltonian cycle of G. Assume therefore that uv lies on C. In this case, we have a Hamiltonian u–v path in G. Since $\deg_G(u) + \deg_G(v) \geq n$, it follows from the Lemma that G is Hamiltonian. ∎