Graphs & Digraphs (Chartrand and Lesniak)

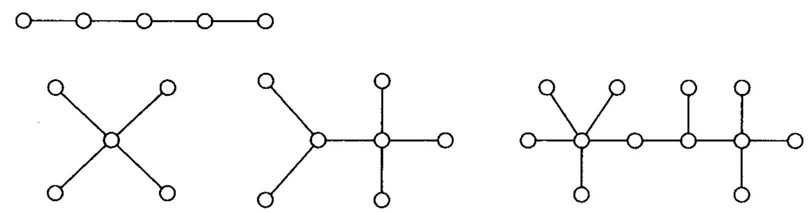


Figure 3.2 Paths, stars, double stars and caterpillars.

Theorem 3.8

Let T be a tree of order k, and let G be a graph with $\delta(G) \geq k-1$. Then G contains a subgraph isomorphic to T.

Proof

We proceed by induction on k. The result is obvious for $k=1$ since K_1 is a subgraph of every graph and for $k=2$ since K_2 is a subgraph of every nonempty graph.

Assume for each tree T' of order $k-1$, $k \geq 3$, and every graph H with $\delta(H) \geq k-2$ that H contains a subgraph isomorphic to T'. Let T be a tree of order k and let G be a graph with $\delta(G) \geq k-1$. We show that G contains a subgraph isomorphic to T.

Let v be an end-vertex of T and let u be the vertex of T adjacent to v. The graph $T-v$ is necessarily a tree of order $k-1$. The graph G has $\delta(G) \geq k-1 > k-2$; thus by the inductive hypothesis, G contains a subgraph F isomorphic to $T-v$. Let u' denote the vertex of F that corresponds to u. Since $\deg_G u' \geq k-1$ and $T-v$ has order $k-1$, the vertex u' is adjacent to a vertex of G that does not belong to F. Therefore, G contains a subgraph isomorphic to T. □

Although no convenient closed formula is known for the number of nonisomorphic trees of order n, a formula does exist for the number of distinct labeled trees (whose vertices are labeled from a fixed set of cardinality n). For $n=3$ and $n=4$, the answer is sufficiently simple that we can actually draw all three distinct trees of order 3 whose vertices are labeled with elements of the set $\{1,2,3\}$ and all 16 distinct trees of order 4 whose vertices are labeled with elements of the set $\{1,2,3,4\}$. These are shown in Figure 3.3.

In general, the number of distinct trees of order n whose vertices are labeled with the same set of n labels is n^{n-2}. This result is due to Cayley [C3]. There have been a number of proofs of Cayley's theorem. The one that we describe here is due to Prüfer [P5]. The proof consists of showing the existence of a one-to-one correspondence between the trees of order n whose vertices are labeled with

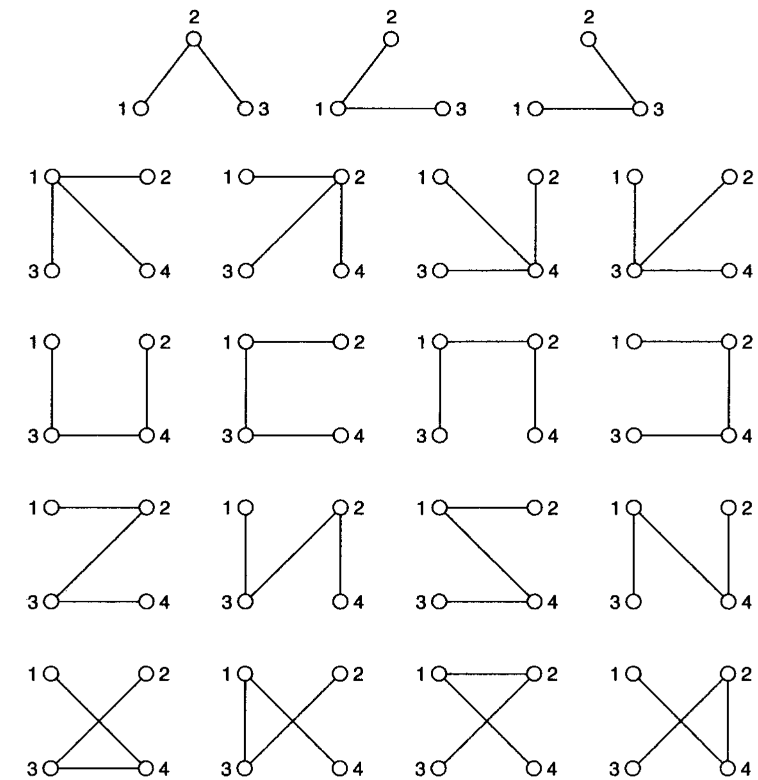
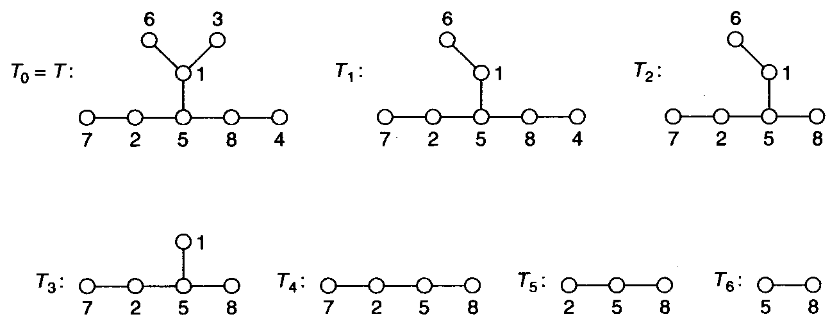


Figure 3.3 The labeled trees of orders 3 and 4.

elements of the set $\{1,2,\ldots,n\}$ and the sequences (called *Prüfer sequences*) of length $n-2$ whose entries are from the set $\{1,2,\ldots,n\}$. Since the number of such sequences is n^{n-2}, once the one-to-one correspondence has been established, the proof is complete. □

Before stating Cayley's theorem formally, we illustrate the technique with an example. Consider the tree T of Figure 3.4 of order $n=8$ whose vertices are labeled with elements of $\{1,2,\ldots,8\}$. The end-vertex of $T_0 = T$ having the smallest label is found, its neighbor is the first term of the Prüfer sequence for T, and this end-vertex is deleted, producing a new tree T_1. The neighbor of the end-vertex of T_1 having the smallest label is the second term of the Prüfer sequence for T; this end-vertex is deleted, producing the tree T_2. We continue this until we arrive at $T_{n-2} = K_2$. The resulting sequence of length $n-2$ is the Prüfer sequence for T.

In the example just described, observe that every vertex v of T appears in its Prüfer sequence $\deg v - 1$ times. This is true in general. Therefore, no end-vertex



Prüfer sequence for T: (1, 8, 1, 5, 2, 5)

Figure 3.4 Determining the Prüfer sequence of a tree.

of T appears in the Prüfer sequence for T. So, if T is a tree of order n and size m, then the number of terms in its Prüfer sequence is

$$\sum_{v \in V(T)} (\deg v - 1) = 2m - n = 2(n-1) - n = n - 2.$$

We now consider the converse question, that is, if $(a_1, a_2, \ldots, a_{n-2})$ is a sequence of length $n-2$ such that each $a_i \in \{1, 2, \ldots, n\}$, then we construct a labeled tree T of order n such that the given sequence is the Prüfer sequence for T. Suppose that we are given the sequence (1, 8, 1, 5, 2, 5). We determine the smallest element of the set $\{1, 2, \ldots, 8\}$ not appearing in this sequence. This element is 3. In T, we join 3 to 1 (the first element of the sequence). The first term is deleted and the reduced sequence (8, 1, 5, 2, 5) is now considered. Also, the element 3 is deleted from the set $\{1, 2, \ldots, 8\}$, and the smallest element of this set not appearing in (8, 1, 5, 2, 5) is found, which is 4, and is joined to 8. This procedure is continued until two elements of the set remain. These two vertices are joined and T is constructed. This is illustrated in Figure 3.5.

Since a step in the second procedure is simply the reverse of a step in the first procedure, we have the desired one-to-one correspondence. We have now described a technique for proving of Cayley's theorem.

Theorem 3.9

There are n^{n-2} distinct labeled trees of order n.

Theorem 3.9 might be considered as a formula for determining the number of distinct spanning trees in the labeled graph K_n. We now consider the same question for graphs in general.

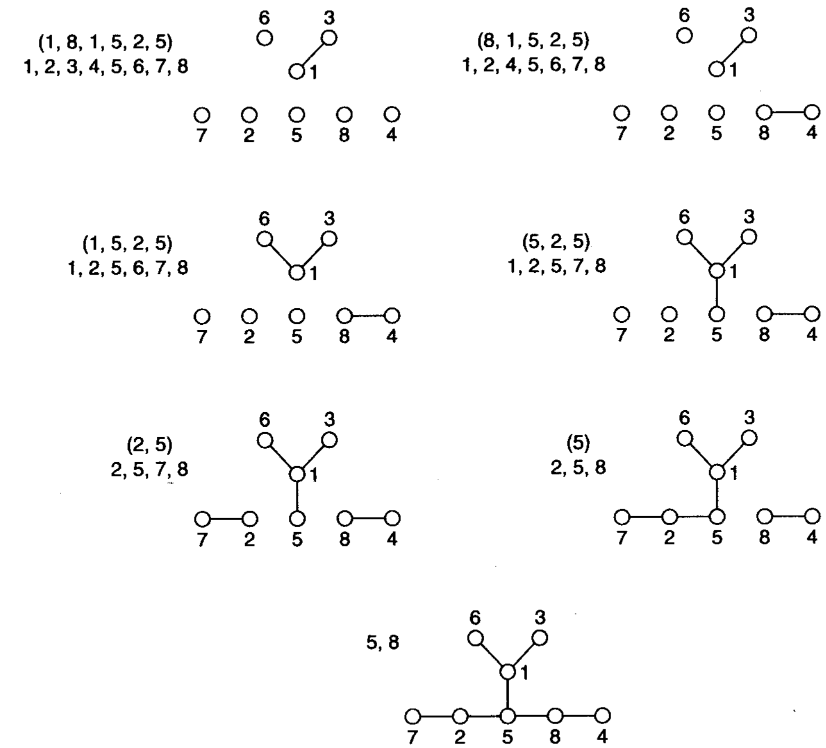


Figure 3.5 Constructing a tree with a given Prüfer sequence.

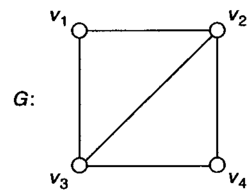
The next result, namely Theorem 3.10, is due to Kirchhoff [K4] and is often referred to as the *Matrix-Tree Theorem*. Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. The *degree matrix* $D(G) = [d_{ij}]$ in the $n \times n$ matrix with $d_{ii} = \deg v_i$ and $d_{ij} = 0$ for $i \neq j$. We now state the Matrix-Tree Theorem.

Theorem 3.10

If G is a nontrivial labeled graph with adjacency matrix A and degree matrix D, then the number of distinct spanning trees of G is the value of any cofactor of the matrix $D - A$.

We illustrate the Matrix-Tree Theorem for the graph G of Figure 3.6, where the matrixes D and $D - A$ are also shown.

To calculate a cofactor of $D - A$, we delete the entries in row i and column j for some i and j with $1 \leq i, j \leq 4$ and multiply $(-1)^{i+j}$ and the determinant of the



$$D: \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \qquad D-A = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

Figure 3.6 Illustrating the Matrix-Tree Theorem.

resulting submatrix. For example, the cofactor of the (2, 3) entry in the matrix $D-A$ in Figure 3.6 is

$$(-1)^{2+3}\begin{vmatrix} 2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & 2 \end{vmatrix}$$

Expanding by the first row, we obtain

$$-\left(2\begin{vmatrix} -1 & -1 \\ -1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} + 0\begin{vmatrix} -1 & -1 \\ 0 & -1 \end{vmatrix}\right)$$
$$= -(2(-3) + 1(-2) + 0) = 8$$

Consequently, there are eight distinct spanning trees of the graph G of Figure 3.6, all of which are shown in Figure 3.7.

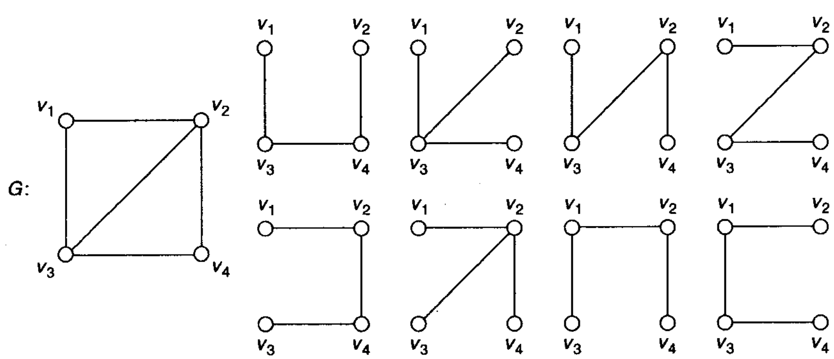


Figure 3.7 The distinct spanning trees of a graph.

EXERCISES 3.1

3.1 Draw all forests of order 6.

3.2 Prove that a graph G is a forest if and only if every induced subgraph of G contains a vertex of degree at most 1.

3.3 Characterize those graphs with the property that every connected subgraph is an induced subgraph.

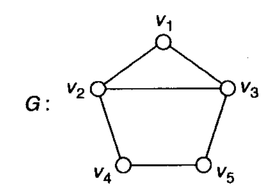
3.4 A tree is called *central* if its center is K_1 and *bicentral* if its center is K_2. Show that every tree is central or bicentral.

3.5 Determine the Prüfer sequences of the trees in Figure 3.3.

3.6 (a) Which trees have constant Prüfer sequences?
(b) Which trees have Prüfer sequences each term of which is one of two numbers?
(c) Which trees have Prüfer sequences with distinct terms?

3.7 Determine the labeled tree having Prüfer sequence (4, 5, 7, 2, 1, 1, 6, 6, 7).

3.8 Let G be the labeled graph below.



(a) Use the Matrix-Tree Theorem to compute the number of distinct labeled spanning trees of G.
(b) Draw all the distinct labeled spanning trees of G.

3.9 (a) Let $G = K_4$ with $V(G) = \{v_1, v_2, v_3, v_4\}$. Draw all spanning trees of G in which v_4 is an end-vertex.
(b) Let v be a fixed vertex of $G = K_n$. Determine the number of spanning trees of G in which v is an end-vertex.

3.10 Prove Theorem 3.9 as a corollary to Theorem 3.10.

3.2 ARBORICITY AND VERTEX-ARBORICITY

One of the most common problems in graph theory deals with decomposition of a graph into various subgraphs possessing some prescribed property. There are ordinarily two problems of this type, one dealing with a decomposition of the vertex set and the other with a decomposition of the edge set. One such property