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Applied Mathematics
Department of Mathematical Sciences

Study Guide

Partial Differential Equations 2018
(10643-AM774 / 20753-AMB834)

Contents

1 General Information

1.1 Objectives

This module is an introduction to the theory and applications of partial differential equations. The main objectives of the module are

- To give the student an elementary introduction to the theory of partial differential equations and to introduce the student to the relevant solution techniques such as the method of characteristics, separation of variables and the use of transform techniques.
- To give the student some skill in elementary continuous modelling. This comprises the mathematical formulation of problems that can be posed as boundary value problems.

Other skills that will be developed during this course are

- A deeper understanding of calculus: The student's skills in manipulation and interpretation of (particularly multivariate) calculus should improve.
- Spatial visualization skills will be developed, as well as how to graphically represent solutions in MATLAB or MATHEMATICA.
- Skill in using symbolic manipulation (MATHEMATICA) for solving mathematical problems.

1.2 Study Material

- Notes will be made available on the website for the student to download and print. The lectures follow these notes closely.
- No textbook is prescribed, however, the following book may be of value (has been prescribed before):
  Partial Differential Equations for Scientists and Engineers, STANLEY J. FARLOW, Dover Publications (1993).

1.3 Assessment

You will write two tests. You will also be required to hand in assignments that will be graded.

The final pass mark is calculated as follows:

$$\text{Pass Mark}(\%) = 0.2 \times \text{Assignments}(\%) + 0.4 \times \text{First Test}(\%) + 0.4 \times \text{Second Test}(\%)$$

Dates and times for the tests and due dates for assignments will appear on the web site.

1.4 Contact Information

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1.5 Course Website

http://appliedmaths.sun.ac.za/TW774/

2 SPECIFIC OUTCOMES AND ASSESSMENT CRITERIA

2.1 CAPABILITIES

A student who has successfully completed this module, can:

- Solve a linear second order partial differential equation (pde) on an appropriate (rectangular, U-shaped or circular) domain by means of separation of variables and enforce the boundary/initial conditions, which may involve Fourier series expansion, in order to find a specific solution.
- Solve a linear second order parabolic or hyperbolic pde on an infinite or semi-infinite domain by applying a suitable transform in space, solving the resulting ode, and transforming back.
- Derive a simple linear pde from first principles by constructing an appropriate small element, applying the appropriate physical principles to the element, and letting the size of the element tend to zero.
- Solve a first order linear, semi-linear or quasi-linear pde with Cauchy condition by using parameterisation along characteristics or by using Lagrange's method. The student is also able to find the coordinates of shock points or the boundaries of domains of multiple-valuedness of solutions.
- Write down a finite difference scheme for solving parabolic pdes numerically and derive the stability condition, or write down the system of equations for solving an elliptic pde with finite differences.

2.2 OUTCOMES

2.2.1 Fourier Analysis

Performances	Assessment	Range
Find the Fourier series (both cosine-sine form as well as complex form) of a given function.	Correctly derived, all integrations successfully completed, final expression(s) simplified, and special cases where separate integration is required completed, series expressed in summation form or the first few terms in the series.	Only piecewise functions consisting of straight lines, exp function, sine or cosine functions. No more involved than double integration by parts.
Select appropriate Fourier series from a list, using properties of the function.	Correct use of parity, period, and continuity to choose the item.	Simple piecewise functions only.
Derive, use, or verify some properties of a Fourier series.	Correct derivation and simplification.	Enough hints will be supplied.
Find the Fourier Transform (FT), the Sine Transform or the Cosine Transform of a given function, and express the function in integral representation form using the appropriate inverse transform.	Correctly derived, all integrations successfully completed, final expression(s) simplified.	Only piecewise functions consisting of straight lines, exp function, sine or cosine functions. No more involved than double integration by parts.
Find the FT or IFT of $e^{-b^2x^2}$ (completely) where b is a positive constant.	Correctly derived. Case when $x = 0$ treated separately.	
Find the FT or IFT of $f(x) = e^{-b^2x^2}$ where b is a positive constant, by using the formula for the FT of $e^{-b^2x^2}$	Correct derivation and simplification.	$f(x)$ only a simple function that leads to integration by parts only once, or that can be absorbed into the exponential.
Derive, use, or verify some properties of the Fourier Transform, Sine Transform or Cosine Transform.	Correct derivation and simplification.	Only linearity, the differentiation property, scaling and shifting of the argument.
Solve linear pdes with appropriate initial and boundary conditions by using the appropriate transform in space, solving an ordinary differential equation in time and then applying the appropriate inverse transform in space.	Correct choice of transform, correct application of transform, correct application of boundary conditions, simplification of the final solution, if possible, otherwise solution may be left in integral form.	PDE: Only the *heat* equation, the *wave* equation, as well as possibly other equations for example $u_t = u + u_x + u_{xx}$, or $u_t = u_{xxx}$. DOMAIN: On $x \in (-\infty, \infty)$, or $x \in [0, \infty)$ and $t \in [0, \infty)$. INITIAL CONDITION: Only functions of which the FT, SineT or CosineT can be found easily.

2.2.2 Separation of Variables

Performances	Assessment	Range
Solve the *heat* equation on a rectangular U-shaped domain by separation of variables with appropriate boundary conditions.	Correct application of separation of variables (fully described), correct choice of general function based on physical assumptions, correct enforcement of boundaries, correct Fourier expansion of initial condition, correct final solution.	BOUNDARIES: Fixed temperature at the ends, or with insulated ends, or with one end held at fixed temperature and the other insulated. INITIAL CONDITION: May require Fourier approximation, but simple to integrate or given, or to be selected from list.
Solve the *wave* equation on a rectangular U-shaped domain by separation of variables with appropriate boundary conditions.	Correct application of separation of variables (fully described), correct choice of general function based on physical assumptions, correct enforcement of boundaries, correct Fourier expansion of initial conditions, correct final solution.	BOUNDARIES: Either or both ends free or fixed. INITIAL CONDITION: Either or both the initial profile and the initial velocity may require Fourier approximation, but simple to integrate or given, or to be selected from list.
Solve *Laplace*'s equation on a rectangular domain by separation of variables with appropriate boundary conditions.	Correct application of separation of variables (fully described), correct choice of general function based on physical assumptions, correct enforcement of boundaries, correct Fourier expansion of boundary conditions, correct final solution.	BOUNDARIES: At least two sides with zero boundaries or linear. Other boundaries may require Fourier approximation, but simple to integrate or given, or to be selected from list.
Solve *Laplace*'s equation on a circular domain or part thereof by separation of variables with appropriate boundary conditions.	Correct application of separation of variables (fully described), full description of the solution of Euler's equation, correct choice of general solution based on physical assumptions, correct enforcement of boundaries, correct Fourier expansion of boundary conditions, correct final solution.	DOMAINS: Circular disk or annulus only. BOUNDARIES: Either or both the inner and outer boundaries may require Fourier approximation, but simple to integrate or given, or to be selected from list.

2.2.3 Derivation of PDEs

Performances	Assessment	Range
Derivation of the 1D *heat* equation (in a rod).	Correct assumptions, appropriate sketch, correct choice of element, correct derivation.	Various insulation conditions may be imposed, as well as non-uniformity in thickness, density and/or conductivity. Fick's first law as well as the relevant heat-temperature relationship will be given.
Derivation of the *wave* equation for a string from first principles.	Correct assumptions, appropriate sketch, correct choice of element, correct derivation.	Non-uniformity in density, and/or thickness, and exclusion or inclusion of gravity or other forces may be imposed.
Derivation of the *wave* equation for a membrane (2D).	Correct assumptions, appropriate sketch, correct choice of element, correct derivation.	Non-uniformity in density, and/or thickness, and exclusion or inclusion of gravity or other forces may be imposed.

2.2.4 First order PDEs: characteristics, Lagrange's method

Performances	Assessment	Range
Find the parameterised solution of a semi-linear first order pde with a Cauchy boundary condition using parameterisation along characteristics.	Correct set of odes, correct solution and enforcement of initial conditions, simplification in terms of x and y (if possible). Correct interpretation if required.	Only equations of the type $a(x)u_x + b(y)u_y = c(x, y, u)$, and the solution of the odes will require only simple separation.
Solve a pde that develops multiple valuedness/shocks using parameterising along characteristics. Find the appropriate envelope of the domain of multiple-valuedness, as well as the time and position of the first occurrence of a shock.	Correct parameterised solution of the pde, correct expression for family of characteristics, correct parameterised envelope of characteristics, correct coordinates of the shock point(s).	Only pdes with simple characteristic traces such as lines, parabolas, and exponential curves.
Find the general solution of quasi-linear first order pdes by using Lagrange's method.	Correct odes, correct canonical expressions, correct enforcement of Cauchy condition, simplification (if possible).	Only types where the ode solution can be obtained by simple separation.

2.2.5 Finite difference methods

Performances	Assessment	Range
Derive finite difference formulae for the first, second and/or third derivatives of a function, and derivation of local truncation error.	Correct derivation, correct Taylor series, correct truncation error.	Only central, forward, backward, and also skew difference formulae on an irregular grid.
Write down the finite difference approximation for Laplace's equation, for various kinds of boundaries, and draw up a system of equations for the solution at the nodes.	Correct difference formulae, correct application of symmetry (where required), correct set of equations, correct matrix form.	BOUNDARIES: Dirichlet alone, or partly Dirichlet and partly Neumann DOMAINS: rectangular, L-shaped, partly circular, partly polygonal (Equations never need to be solved by hand.)
Write down the finite difference scheme for solving a linear parabolic equation using appropriate difference approximations for the spatial derivatives and either Euler or Crank-Nicolson time stepping for the time derivative.	Correct scheme, correct enforcement of boundaries, correct starting procedure.	No larger than five point spatial formulae. No higher derivative than third. Simple boundary conditions (zero or constant).
Explain the meaning of 'linear stability' of a numerical scheme, and find linear stability condition.	Explanation, correct mode-growth expression, correct inequality, correct final condition.	In cases where the mode growth expression has a simple trigonometric term in ωh, find the maximum yourself. In cases where the mode growth expression has a complicated expression in ωh, the maximum will be given (or hints on how to find it will be given).