
Problem 1:

(a) $A = 1,\ B = -1,\ C = 3 \implies B^2 - 4AC < 0 \implies$ the PDE is elliptic

(b) $A = 1,\ B = -9,\ C = 0 \implies B^2 - 4AC > 0 \implies$ the PDE is hyperbolic

(c) $A = 1,\ B = 2,\ C = 1 \implies B^2 - 4AC = 0 \implies$ the PDE is parabolic

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Problem 2:

Since $u(0,t) = 0$ and $u(L,t) = 0$, we may use the result derived in Lecture 11:

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-k(\frac{n^2\pi^2}{L^2})t} \sin \tfrac{n\pi x}{L}.$$

As we saw in that lecture, the constants $c_1, c_2, \ldots$ are the coefficients in the sine series expansion of $u(x,0)$:

$$c_n = \tfrac{2}{L}\left[\int_0^{\frac{L}{2}} 1 \cdot \sin \tfrac{n\pi x}{L}\, dx + \int_{\frac{L}{2}}^{L} 0 \cdot \sin \tfrac{n\pi x}{L}\, dx\right] = \tfrac{2}{n\pi}\big(1 - \cos \tfrac{n\pi}{2}\big).$$

Therefore $u(x,t) = \frac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\left[\frac{1-\cos\frac{n\pi}{2}}{n}\right] e^{-k(\frac{n^2\pi^2}{L^2})t} \sin \tfrac{n\pi x}{L}$.

Problem 3:

The boundary value problem to be solved: $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2},\ \ \dfrac{\partial u}{\partial x}(0,t) = 0,\ \ \dfrac{\partial u}{\partial x}(L,t) = 0,\ \ u(x,0) = f(x).$

Substituting $u(x,t) = X(x)T(t)$ into the PDE leads to $X'' + \lambda X = 0$ and $T' + k\lambda T = 0$. The boundary conditions are $X'(0)T(t) = 0$ and $X'(L)T(t) = 0$, which must hold for all t, so we take $X'(0) = 0$ and $X'(L) = 0$.

Thus we have the Sturm-Liouville problem: $X'' + \lambda X = 0,\ \ X'(0) = 0,\ \ X'(L) = 0.$

According to Tutorial 2, Problem 5, the eigenvalues for this problem are $\lambda_n = \frac{n^2\pi^2}{L^2}$ with corresponding non-trivial solutions $X_n = \cos\frac{n\pi x}{L}$, $n = 0, 1, 2, \ldots$.

We solve for T from $T' = -k\lambda T$: $T_n = e^{-k\lambda_n t} = e^{-k(\frac{n^2\pi^2}{L^2})t}$, $n = 0, 1, 2, \ldots$.

We note that $X_0T_0 = 1$, and use the superposition principle to form $u(x,t) = c_0 + \displaystyle\sum_{n=1}^{\infty} c_n e^{-k(\frac{n^2\pi^2}{L^2})t}\cos\tfrac{n\pi x}{L}$.

The condition $u(x,0) = f(x)$ implies $f(x) = c_0 + \displaystyle\sum_{n=1}^{\infty} c_n \cos\tfrac{n\pi x}{L}$, which is the cosine series expansion of $f(x)$.

turn over...

The coefficients are $c_0 = \frac{a_0}{2} = \frac{1}{L}\displaystyle\int_0^L f(x)\,dx$ and $c_n = a_n = \frac{2}{L}\displaystyle\int_0^L f(x)\,\cos\frac{n\pi x}{L}\,dx$, so that

$$u(x,t) = \frac{1}{L}\int_0^L f(x)\,dx + \frac{2}{L}\sum_{n=1}^{\infty}\left[\int_0^L f(x)\,\cos\frac{n\pi x}{L}\,dx\right] e^{-k(\frac{n^2\pi^2}{L^2})t}\cos\frac{n\pi x}{L}.$$

We note that $\lim\limits_{t\to\infty} u(x,t) = \frac{1}{L}\displaystyle\int_0^L f(x)\,dx$, which is the average of the initial temperature distribution through the rod. It makes sense, since no heat flows in or out of the insulated ends.

Problem 4:

(a) Under the assumption $u(x,y) = X(x)Y(y)$, the PDE becomes $XY' = XY' \implies \frac{X'}{X} = \frac{Y'}{Y} = -\lambda$, so that $X' = -\lambda X$ and $Y' = -\lambda Y$, where λ is a constant.

We solve these two ordinary DEs: $X = k_1 e^{-\lambda x}$ and $Y = k_2 e^{-\lambda y}$,

and form the solution $\boxed{u(x,y) = c_1 e^{c_2(x+y)}}$ (with constants $c_1 = k_1k_2$ and $c_2 = -\lambda$).

(b) $X'Y + XY' = XY \implies X(Y - Y') = X'Y \implies \frac{X'}{X} = \frac{Y-Y'}{Y} = -\lambda.$

$X' = -\lambda X \implies X = k_1 e^{-\lambda x}$, and $Y - Y' = -\lambda Y \implies Y' = (1+\lambda)Y \implies Y = k_2 e^{(1+\lambda)y}$

Therefore $\boxed{u(X,y) = c_1 e^{c_2(x-y)+y}}$ (with constants $c_1 = k_1k_2$ and $c_2 = -\lambda$).

(c) $xX'Y = yXY' \implies \frac{xX'}{X} = \frac{yY'}{Y} = -\lambda$

$x\frac{dX}{dx} = -\lambda X \implies \displaystyle\int \frac{1}{X}\,dX = -\lambda\int \frac{1}{x}\,dx \implies \ln|X| = -\lambda\ln|x| + c' \implies X = k_1 x^{-\lambda}$

Similarly, $Y = k_2 y^{-\lambda}$, hence $\boxed{u(x,y) = c_1(xy)^{c_2}}$ (with constants $c_1 = k_1k_2$ and $c_2 = -\lambda$)
