
Problem 1:

$f(x) = |\sin x|$ is even, so we determine its cosine series:

$a_0 = \frac{2}{\pi}\int_0^\pi \sin x\,dx = \frac{4}{\pi}, \quad a_1 = \frac{2}{\pi}\int_0^\pi \sin x\,\cos x\,dx = 0, \quad a_n = \frac{2}{\pi}\int_0^\pi \sin x\,\cos nx\,dx = \frac{2(-1)^n+2}{\pi(1-n^2)}, \quad n = 2,3,\ldots$

Therefore $f(x) = \frac{2}{\pi} + \frac{2}{\pi}\sum_{n=2}^{\infty} \frac{(-1)^n+1}{1-n^2}\cos nx.$

Problem 2:

We find the sine series of $w(x)$: $b_n = \frac{2}{L}\int_0^L \frac{w_0}{L}x\,\sin\frac{n\pi x}{L}\,dx = \frac{2w_0(-1)^{n+1}}{\pi n} \quad\Longrightarrow\quad w(x) = \frac{2w_0}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{L}.$

The differential equation can then be written as $k\frac{d^4y}{dx^4} = \frac{2w_0}{\pi}\sum_{n=1}^{\infty}\frac{(-1^{n+1})}{n}\sin\frac{n\pi x}{L}.$

As a particular solution we try $y_p = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$, and substitute that into the DE to solve for B_n:

$k\sum_{n=1}^{\infty} B_n\left(\frac{n\pi}{L}\right)^4 \sin\frac{n\pi x}{L} = \frac{2w_0}{\pi}\sum_{n=1}^{\infty}\frac{(-1^{n+1})}{n}\sin\frac{n\pi x}{L} \quad\Longrightarrow\quad B_n = \frac{2w_0L^4(-1)^{n+1}}{k\pi^5n^5}.$

Hence $y_p = \frac{2w_0L^4}{k\pi^5}\sum_{n+1}^{\infty}\frac{(-1)^{n+1}}{n^5}\sin\frac{n\pi x}{L}.$

Problem 3:

$f(x) = x\pi^2 - x^3, \;\; f'(x) = \pi^2 - 3x^2, \;\; f''(x) = -6x.$

We observe that both f and f' (and their periodic extensions) are continuous, but the periodic extension of f'' is discontinuous, so we expect b_n to be asymptotically equivalent to c/n^3 as $n \to \infty$.

Problem 4:

Using the results from Problem 1 above, we note that $a_n = 0$ for n odd, and $a_n = \frac{4}{\pi(1-n^2)}$ for n even.

Parseval's theorem:

$$\begin{aligned}
\frac{1}{\pi}\int_{-\pi}^{\pi}\sin^2 x\,dx &= \frac{1}{2}\left(\frac{4}{\pi}\right)^2 + \sum_{n=2,4,6,\ldots}\left(\frac{4}{\pi(1-n^2)}\right)^2 \\
1 &= \frac{8}{\pi^2} + \frac{16}{\pi^2}\sum_{n=2,4,6,\ldots}\frac{1}{(1-n)^2(1+n)^2} \\
\sum_{n=2,4,6,\ldots}\frac{1}{(1-n)^2(1+n)^2} &= \frac{\pi^2}{16}\left(1 - \frac{8}{\pi^2}\right) \\
\frac{1}{1^2.3^2} + \frac{1}{3^2.5^2} + \frac{1}{5^2.7^2} + \cdots &= \frac{\pi^2}{16} - \frac{1}{2}.
\end{aligned}$$

Problem 5:

Suppose $\lambda = 0$: $y'' = 0 \implies y = c_1 x + c_2$

$y' = c_1$

$y'(0) = 0: \; c_1 = 0$

$y'(L) = 0: \; c_1 = 0$

Hence $y = c_2$ is a solution, which is non-trivial as long as $c_2 \neq 0$. Thus $\lambda = 0$ is an eigenvalue.

Suppose $\lambda < 0$, say $\lambda = -\alpha^2$: $y'' = \alpha^2 y \implies y = c_1 \cosh \alpha x + c_2 \sinh \alpha x$

$y' = c_1 \alpha \sinh \alpha x + c_2 \alpha \cosh \alpha x$

$y'(0) = 0: \; c_1\alpha \cdot 0 + c_2 \alpha \cdot 1 = 0 \implies c_2 = 0$

$y'(L) = 0: \; c_1 \alpha \sinh \alpha L + 0 = 0 \implies c_1 = 0 \quad [\sinh \alpha L = 0 \text{ only if } \alpha L = 0, \text{ but } \alpha L \neq 0]$

Hence $y = 0$ (the trivial solution) is the only possible solution when $\lambda < 0$.

Suppose $\lambda > 0$, say $\lambda = \alpha^2$: $y'' = -\alpha^2 y \implies y = c_1 \cos \alpha x + c_2 \sin \alpha x$

$y' = -c_1 \alpha \sin \alpha x + c_2 \alpha \cos \alpha x$

$y'(0) = 0: \; -c_1 \alpha \cdot 0 + c_2 \alpha \cdot 1 = 0 \implies c_2 = 0$

$y'(L) = 0: \; -c_1 \alpha \sin \alpha L + 0 = 0 \implies c_1 = 0 \text{ or } \alpha = \frac{n\pi}{L}, \; n \in \mathbb{Z}$

By choosing $\alpha = \frac{n\pi}{L}$, that is $\lambda = \frac{n^2\pi^2}{L^2}$, we get non-trivial solutions (as long as $c_1 \neq 0$).

Eigenvalues: $\lambda_n = \frac{n^2\pi^2}{L^2}, \; n = 0, 1, 2, \ldots$.

Eigenfunctions: $y_n = \cos \frac{n\pi x}{L}, \; n = 0, 1, 2, \ldots$.

The eigenfunctions are defined up to scale, so we may omit the constants c_1 and c_2. Also note that $\lambda_0 = 0$ and $y_0 = 1$, which correspond to our findings in the case of $\lambda = 0$ above.