
Problem 1:

(a) $(f_1, f_2) = \int_{-2}^{2} x^3\,dx = \frac{1}{4}x^4\big|_{-2}^{2} = 0.$

(b) $(f_1, f_2) = \int_{0}^{2} (x-1)\,dx = \left[\frac{1}{2}x^2 - x\right]_0^2 = 0.$

(c) $(f_1, f_2) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\,\cos 2x\,dx = \frac{1}{2}x\,\sin 2x\big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin 2x\,dx = 0 + \frac{1}{4}\cos 2x\big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0.$

Problem 2:

Let $\phi_0(x) = 1$, $\phi_n(x) = \cos nx$ and $\psi_n(x) = \sin nx$, $n = 1, 2, \ldots$. It was shown in class (lecture 1) that $(\phi_m, \phi_n) = 0$, $m \neq n$. Now,

- $(\psi_n, \phi_0) = \int_{-\pi}^{\pi} \sin nx\,dx = -\frac{1}{n}\cos nx\big|_{-\pi}^{\pi} = 0,$
- $(\psi_n, \phi_n) = \int_{-\pi}^{\pi} \sin nx\,\cos nx\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin 2nx\,dx = -\frac{1}{4n}\cos 2nx\big|_{-\pi}^{\pi} = 0,$
- $(\psi_m, \phi_n) = \int_{-\pi}^{\pi} \sin mx\,\cos nx\,dx = \frac{1}{2}\left[\frac{\cos(m-n)x}{m-n} + \frac{\cos(m+n)x}{m+n}\right]_{-\pi}^{\pi} = 0, \quad m \neq n,$
- $(\psi_m, \psi_n) = \int_{-\pi}^{\pi} \sin mx\,\sin nx\,dx = \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi} = 0, \quad m \neq n.$

Problem 3:

(a) $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} 1\,dx\right] = 1$

$a_n = \frac{1}{\pi}\int_{0}^{\pi} \cos nx\,dx = \frac{1}{n\pi}\sin nx\big|_0^{\pi} = 0$

$b_n = \frac{1}{\pi}\int_{0}^{\pi} \sin nx\,dx = -\frac{1}{n\pi}\cos nx\big|_0^{\pi} = \frac{1-(-1)^n}{n\pi}$

Therefore $f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n\pi}\sin nx.$

(b) $a_0 = \frac{1}{\pi}\int_{0}^{\pi} x^2\,dx = \frac{1}{3\pi}x^3\big|_0^{\pi} = \frac{\pi^2}{3}$

$a_n = \frac{1}{\pi}\int_{0}^{\pi} x^2\cos nx\,dx = \frac{1}{\pi}\left[\frac{x^2\sin nx}{n} - \frac{2\sin nx}{n^3} + \frac{2x\cos nx}{n^2}\right]_0^{\pi} = \frac{2(-1)^n}{n^2}$

$b_n = \frac{1}{\pi}\int_{0}^{\pi} x^2\sin nx\,dx = \frac{1}{\pi}\left[-\frac{x^2\cos nx}{n} + \frac{2\cos nx}{n^3} + \frac{2x\sin nx}{n^2}\right]_0^{\pi} = 2\frac{(-1)^n - 1}{n^3\pi} - \frac{(-1)^n\pi}{n}$

Therefore $f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^n}{n^2}\cos nx + \left(2\frac{(-1)^n-1}{n^3\pi} - \frac{(-1)^n\pi}{n}\right)\sin nx\right].$

(c) $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)\,dx = \frac{1}{\pi}\left[\frac{1}{2}x^2+\pi x\right]_{-\pi}^{\pi} = 2\pi$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)\cos nx\,dx = \frac{1}{\pi}\left[\frac{\cos nx}{n^2}+\frac{x\sin nx}{n}+\frac{\pi\sin nx}{n}\right]_{-\pi}^{\pi} = 0$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)\sin nx\,dx = \frac{1}{\pi}\left[\frac{\sin nx}{n^2}-\frac{x\cos nx}{n}-\frac{\pi\cos nx}{n}\right]_{-\pi}^{\pi} = \frac{2(-1)^{n+1}}{n}$

Therefore $f(x) = \pi + \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin nx.$

Problem 4:

With f defined as in Problem 3(b), we observe that $x=\pi$ is a point discontinuity in the periodic extension of f. At this point the Fourier series will therefore converge to the average of $f(\pi^-)=\pi^2$ and $f(\pi^+)=0$, which is $\frac{\pi^2}{2}$. Then, by setting $x=\pi$ in the Fourier series of f, we get

$$\begin{aligned}
\frac{\pi^2}{2} &= \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^n}{n^2}\cos n\pi + \left(2\frac{(-1)^n-1}{n^3\pi}-\frac{(-1)^n\pi}{n}\right)\sin n\pi\right] \\
&= \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^{2n}}{n^2}+0\right] \\
\frac{\pi^2}{2}-\frac{\pi^2}{6} &= 2\sum_{n=1}^{\infty}\frac{1}{n^2} \\
\frac{2\pi^2}{6} &= 2\sum_{n=1}^{\infty}\frac{1}{n^2} \\
\frac{\pi^2}{6} &= 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots.
\end{aligned}$$
