
Problem 1:

(a) The following was derived in Lecture 12 (where we have already taken $g(x) = 0$, so that $B_n = 0$):

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos\tfrac{an\pi t}{L} \sin\tfrac{n\pi x}{L}, \qquad \text{with} \;\; A_n = \tfrac{2}{L}\int_0^L f(x)\sin\tfrac{n\pi x}{L}\,dx.$$

With $L = 1$ and $f(x)$ as given in the assignment, we arrive at

$$A_n = \frac{2\sin(\alpha n\pi)}{\alpha(1-\alpha)n^2\pi^2} \quad \Longrightarrow \quad u(x,t) = \sum_{n=1}^{\infty}\left[\frac{2\sin(\alpha n\pi)}{\alpha(1-\alpha)n^2\pi^2}\right]\cos(an\pi t)\sin(n\pi x).$$

(b) We saw in Lecture 13 that the n-th harmonic has amplitude $|C_n \sin\frac{n\pi x}{L}|$, which reduces to $|A_n \sin(n\pi x)|$ for this particular case. So, for the n-th harmonic to be zero over all x, A_n must be zero. We see that $A_n = 0$ whenever αn is integer (as long as $\alpha \neq 0$ and $\alpha \neq 1$). Consequently, the n-th harmonic is zero for $\alpha \in \{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}\}$.

Problem 2:

We use the expressions on the last page of Lecture 15, with $a = 1$, $b = 1$, $f(x) = 0$, $g(x) = \sin(\frac{5\pi}{2}x)$, $F(y) = 0$ and $G(y) = y$ to determine the coefficients:

$$A_n = 0, \quad B_n = \frac{8n(-1)^n}{\pi\sinh(n\pi)(25-4n^2)}, \quad C_n = 0, \quad D_n = \frac{2(-1)^{n+1}}{n\pi\sinh(n\pi)},$$

which leads to the solution $u(x,y) = u_1(x,y) + u_2(x,y)$:

$$u(x,y) = \sum_{n=1}^{\infty}\left[\frac{8n(-1)^n}{\pi\sinh(n\pi)(25-4n^2)}\right]\sinh(n\pi y)\sin(n\pi x) + \sum_{n=1}^{\infty}\left[\frac{2(-1)^{n+1}}{n\pi\sinh(n\pi)}\right]\sinh(n\pi x)\sin(n\pi y).$$