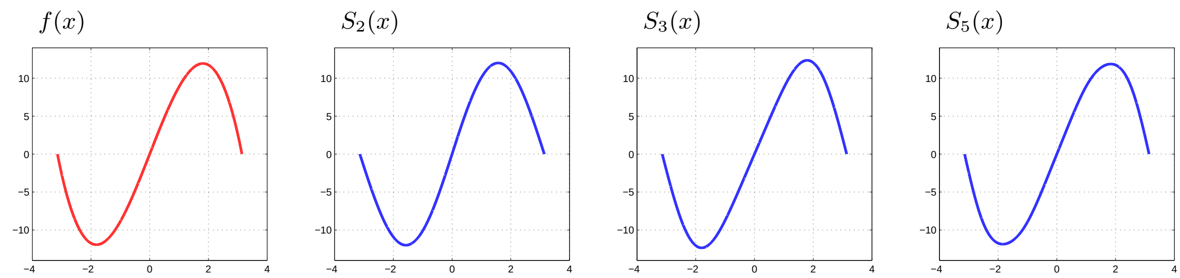

Problem 1:

(a) $b_n = \frac{2}{\pi}\int_0^\pi (\pi^2 x - x^3)\sin nx\,dx = 2\pi\int_0^\pi x\sin nx\,dx - \frac{2}{\pi}\int_0^\pi x^3\sin nx\,dx = 12\frac{(-1)^{n+1}}{n^3}$

Hence $f(x) = 12\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}\sin nx.$

(b)



The partial sum approximations do indeed converge to $f(x)$ at a very rapid rate.

(c)

$$\int x^2\,dx = \frac{\pi^2}{3}\int 1\,dx + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\int\cos nx\,dx$$

$$\tfrac{1}{3}x^3 = \frac{\pi^2}{3}x + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin nx + c$$

Let $x = 0:$ $$\tfrac{1}{3}0^3 = \frac{\pi^2}{3}0 + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}0 + c \implies c = 0$$

$$\pi^2 x - x^3 = 12\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}\sin nx$$

Problem 2:

(a) Suppose $\lambda = 0$: $y'' = 0 \implies y = c_1 x + c_2$ and $y' = c_1$

$y(0) + y'(0) = 0:\ c_1 + c_2 = 0$

$y(1) = 0:\ c_1 + c_2 = 0 \implies c_2 = -c_1$

Hence $\lambda = 0$ is indeed an eigenvalue, with corresponding eigenfunction $y = x - 1$.

Suppose $\lambda < 0$, say $\lambda = -\alpha^2$: $y'' = \alpha^2 y \implies y = c_1\cosh\alpha x + c_2\sinh\alpha x$

$y' = c_1\alpha\sinh\alpha x + c_2\alpha\cosh\alpha x$

$y(0) + y'(0) = 0:\ c_1\cdot 1 + c_2\cdot 0 + c_1\alpha\cdot 0 + c_2\alpha\cdot 1 = 0 \implies c_1 = -\alpha c_2$

$y(1) = 0:\ c_1\cosh\alpha + c_2\sinh\alpha = 0$

$c_2(\sinh\alpha - \alpha\cosh\alpha) = 0$

Note that $\sinh\alpha - \alpha\cosh\alpha \neq 0$ for any $\alpha \neq 0$, so c_2 must be 0, and $c_1 = -\alpha c_2 = 0$.

Hence $y = 0$ is the only possible solution when $\lambda < 0$.

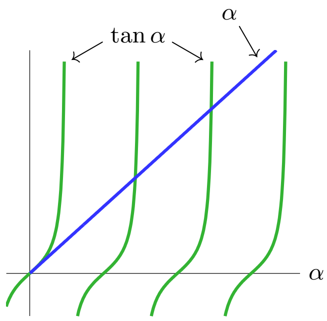
Suppose $\lambda > 0$, say $\lambda = \alpha^2$: $y'' = -\alpha^2 y \implies y = c_1 \cos\alpha x + c_2 \sin\alpha x$

$$y' = -c_1\alpha \sin\alpha x + c_2\alpha\cos\alpha x$$

$$y(0) + y'(0) = 0: \; c_1 \cdot 1 + c_2 \cdot 0 - c_1\alpha \cdot 0 + c_2\alpha \cdot 1 = 0 \implies c_1 = -\alpha c_2$$

$$y(1) = 0: \; c_1 \cos\alpha + c_2 \sin\alpha = 0$$

$$c_2(\sin\alpha - \alpha\cos\alpha) = 0$$



To avoid $c_2 = 0$ (which would lead to $c_1 = 0$ and then $y = 0$) we must choose α for which $\sin\alpha - \alpha\cos\alpha = 0$, or $\alpha = \tan\alpha$. The graphs to the right indicate that this equation has infinitely many solutions, and we use a numerical solver to find the 3 smallest positive solutions:

$\alpha_1 = 4.493409\ldots$

$\alpha_2 = 7.725252\ldots$

$\alpha_3 = 10.90412\ldots$

The 4 smallest eigenvalues are then

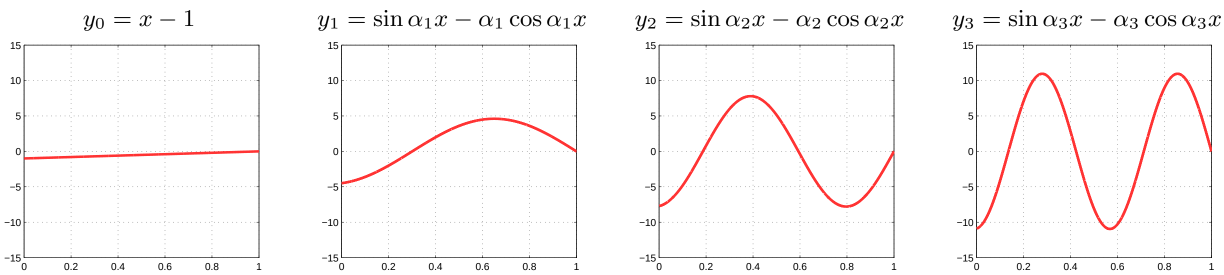
$\lambda_0 = 0$

$\lambda_1 = 20.19073\ldots$

$\lambda_2 = 59.67952\ldots$

$\lambda_3 = 118.8999\ldots$

Plots of the eigenfunctions are shown below (c_2 is arbitrarily chosen as 1).



(b) We are looking for solutions to $\alpha = \tan\alpha$. We can see from the graphs of α and $\tan\alpha$ above that for large α, the points of intersection are close to the positive asymptotes of $\tan\alpha$. We conclude that $\alpha_n \approx \frac{1}{2}(2n+1)\pi$.

Therefore $\lambda_{500} = (\alpha_{500})^2 \approx (\frac{1001}{2}\pi)^2 = 2\,472\,338$.