TW364 Lecture 21

The sine and cosine transforms are not suitable for transforming the <u>first derivative</u> (or any <u>odd</u> derivative).

I'll leave it as an exercise to show that

$$\mathcal{F}_s\{f'(x)\} = -\alpha \mathcal{F}_c\{f(x)\},$$

$$\mathcal{F}_c\{f'(x)\} = \alpha \mathcal{F}_s\{f(x)\} - f(0).$$

$\Rightarrow$ The transform of $f'(x)$ is not expressed in terms of the same transform of $f(x)$.

<u>Sine transform of a second derivative</u>

Suppose f and f' are continuous on $[0,\infty)$, f is abs. integrable, and also that f'' is piecewise continuous on every finite interval,

$f \to 0$ and $f' \to 0$ as $x \to \infty$.

$$\begin{aligned}\mathcal{F}_s\{f''(x)\} &= \int_0^\infty f''(x)\sin(\alpha x)\,dx \\ &= f'(x)\sin(\alpha x)\Big|_0^\infty - \alpha\int_0^\infty f'(x)\cos(\alpha x)\,dx \\ &= -\alpha\left[f(x)\cos(\alpha x)\Big|_0^\infty + \alpha\int_0^\infty f(x)\sin(\alpha x)\,dx\right] \\ &= \alpha f(0) - \alpha^2\mathcal{F}_s\{f(x)\}\end{aligned}$$

$$\therefore \boxed{\mathcal{F}_s\{f''(x)\} = -\alpha^2 \mathcal{F}_s\{f(x)\} + \alpha f(0)}$$

Cosine transform of a second derivative

Under the same conditions,

$$\mathcal{F}_c\{f''(x)\} = \int_0^\infty f''(x)\cos(\alpha x)\,dx$$

$$= f'(x)\cos(\alpha x)\Big|_0^\infty + \alpha\int_0^\infty f'(x)\sin(\alpha x)\,dx$$

$$= -f'(0) + \alpha\left[f(x)\sin(\alpha x)\Big|_0^\infty - \alpha\int_0^\infty f(x)\cos(\alpha x)\,dx\right]$$

$$= -f'(0) - \alpha^2\mathcal{F}_c\{f(x)\}$$

$$\therefore \boxed{\mathcal{F}_c\{f''(x)\} = -\alpha^2\mathcal{F}_c\{f(x)\} - f'(0)}$$

Using transforms to solve BVPs

We may use a Fourier / sine / cosine transform to eliminate one variable, and turn the PDE into an ODE.

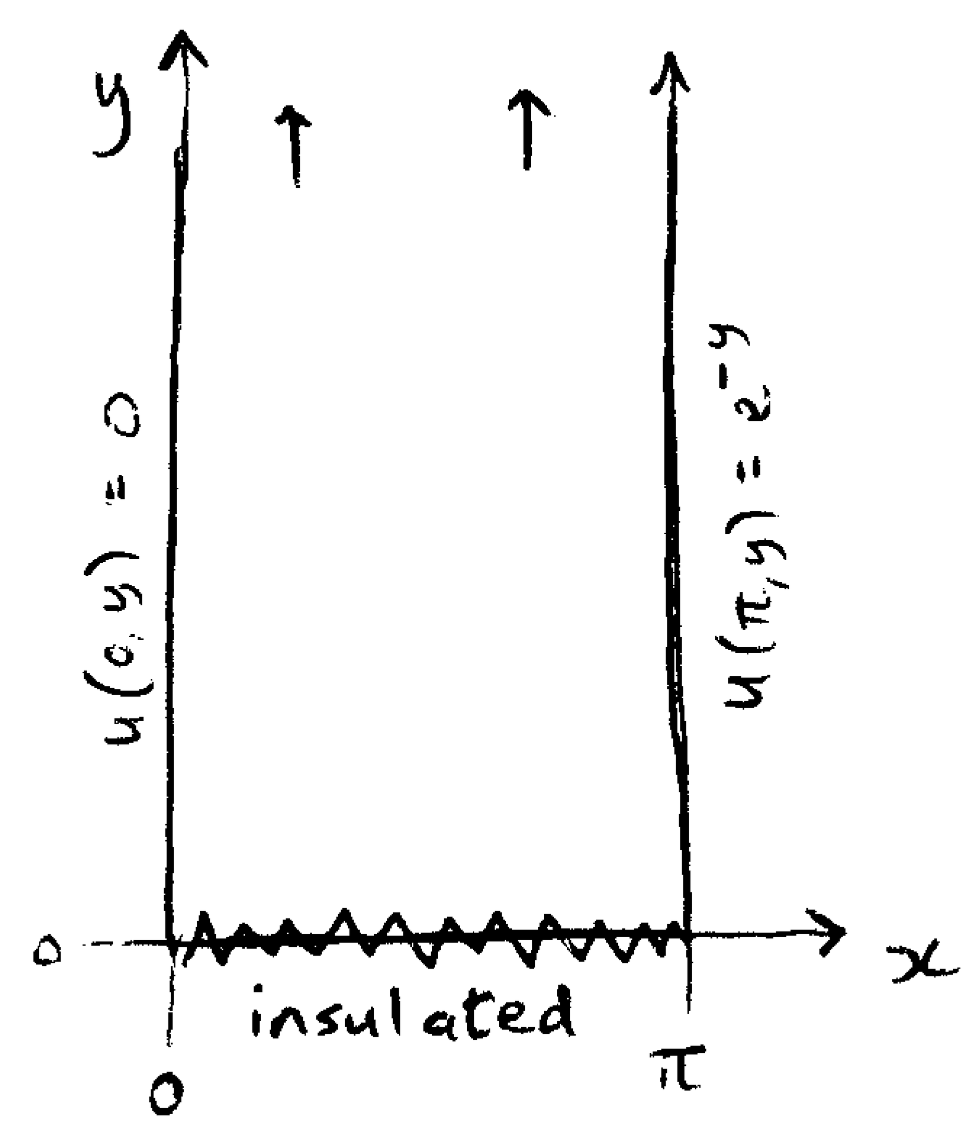
If the domain of that variable is $(-\infty, \infty)$, use a Fourier transform.

If the domain is $[0, \infty)$, and ...

* the boundary condition is a function value, use a sine transform.
* the boundary condition is a derivative, use a cosine transform.

Example

Find the steady-state temperature in a semi-infinite plate:



$$\text{solve} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \begin{matrix} 0 < x < \pi \\ 0 < y < \infty \end{matrix}$$

subject to

$$u(0,y) = 0 \quad , \quad y > 0$$

$$u(\pi,y) = e^{-y} \quad , \quad y > 0$$

$$u_y(x,0) = 0 \quad , \quad 0 < x < \pi$$

The domain of y and the condition on its derivative at $y=0$ indicate the suitability of the cosine transform.

Let $U(x,\alpha) = \mathcal{F}_c\{u(x,y)\}$ [the transform w.r.t. y]

Then $\mathcal{F}_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \frac{d^2 U}{dx^2}$

$$\mathcal{F}_c\left\{\frac{\partial^2 u}{\partial y^2}\right\} = -\alpha^2 U - u_y(x,0)$$

$$\therefore \quad \frac{d^2 U}{dx^2} - \alpha^2 U - u_y(x,0) = 0 \quad \Rightarrow \quad \frac{d^2 U}{dx^2} = \alpha^2 U$$

Therefore $U(x,\alpha) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x)$

Transform the boundary conditions :

$$u(0,y) = 0 \; : \; \mathcal{F}_c\{u(0,y)\} = \mathcal{F}_c\{0\} \;\Rightarrow\; U(0,\alpha) = 0$$

$$u(\pi,y) = e^{-y} \; : \; \mathcal{F}_c\{u(\pi,y)\} = \mathcal{F}_c\{e^{-y}\} \;\Rightarrow\; U(\pi,\alpha) = \frac{1}{1+\alpha^2} \quad \boxed{*}$$

Enforce these conditions :

$$U(0,\alpha) = 0 \; : \; c_1 \cdot 1 + c_2 \cdot 0 = 0 \;\Rightarrow\; c_1 = 0$$

$$U(\pi,\alpha) = \frac{1}{1+\alpha^2} \; : \; c_2 \cdot \sinh(\alpha\pi) = \frac{1}{1+\alpha^2} \;\Rightarrow\; c_2 = \frac{1}{(1+\alpha^2)\sinh(\alpha\pi)}$$

$$\therefore \; U(x,\alpha) = \frac{\sinh(\alpha x)}{(1+\alpha^2)\sinh(\alpha\pi)}$$

Take the inverse transform :

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \frac{\sinh(\alpha x)}{(1+\alpha^2)\sinh(\alpha x)} \cos(\alpha y)\, d\alpha \, .$$

$$\boxed{*} \quad \mathcal{F}_c\{e^{-y}\} = \int_0^\infty e^{-y}\cos(\alpha y)\, dy = \frac{1}{1+\alpha^2} \quad \text{(using integration by parts)}$$