# 14.4 FOURIER TRANSFORMS

Prompted by the complex Fourier integral (Lecture 19), we define the Fourier transform pair:

**Fourier transform:**

$$\mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx = F(\alpha)$$

**Inverse Fourier transform:**

$$\mathcal{F}^{-1}\{F(\alpha)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\, e^{-i\alpha x}\, d\alpha = f(x)$$

... and, prompted by the sine and cosine integrals, we define:

**Sine transform:** $\mathcal{F}_s\{f(x)\} = \int_0^{\infty} f(x)\sin(\alpha x)\, dx = F(\alpha)$

**Inverse sine transform:** $\mathcal{F}_s^{-1}\{F(\alpha)\} = \frac{2}{\pi}\int_0^{\infty} F(\alpha)\sin(\alpha x)\, d\alpha = f(x)$

**Cosine transform:** $\mathcal{F}_c\{f(x)\} = \int_0^{\infty} f(x)\cos(\alpha x)\, dx = F(\alpha)$

**Inverse cosine transform:** $\mathcal{F}_c^{-1}\{F(\alpha)\} = \frac{2}{\pi}\int_0^{\infty} F(\alpha)\cos(\alpha x)\, d\alpha = f(x)$

## Fourier transform of a derivative

Suppose $f$ is continuous and absolutely integrable on $(-\infty, \infty)$, and $f'$ is piecewise continuous.

Also, suppose $f(x) \to 0$ as $x \to \pm\infty$.

$$\mathcal{F}\{f'(x)\} = \int_{-\infty}^{\infty} f'(x)\, e^{i\alpha x}\, dx$$

$$= f(x)\, e^{i\alpha x} \Big|_{-\infty}^{\infty} - i\alpha \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx$$

$$= 0 - i\alpha\, \mathcal{F}\{f(x)\}$$

$$\therefore \boxed{\mathcal{F}\{f'(x)\} = -i\alpha\, \mathcal{F}\{f(x)\}}$$

Similarly, if $f'$ is continuous, $f''$ is piecewise continuous, and $f'(x) \to 0$ as $x \to \pm\infty$,

$$\boxed{\mathcal{F}\{f''(x)\} = -\alpha^2\, \mathcal{F}\{f(x)\}}$$

<u>Note</u>: in the examples that follow, we shall assume $u$ and its derivatives all approach zero as the independent variable approaches $\pm\infty$.

For most applications, this condition won't be a major restriction.

## Example

Solve the heat equation : $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ , $-\infty < x < \infty$ , $t > 0$

Subject to : $u(x,t) \to 0$ as $x \to \pm\infty$

$$u(x,0) = \begin{cases} u_0, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

In this case, we're looking to solve the heat equation over an infinite rod.

Let $U(\alpha,t) = \mathcal{F}\{u(x,t)\}$

Then $\mathcal{F}\{\frac{\partial u}{\partial t}\} = \int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{i\alpha x} dx = \frac{d}{dt}\int_{-\infty}^{\infty} u e^{i\alpha x} dx = \frac{dU}{dt}$

$\mathcal{F}\{k \frac{\partial^2 u}{\partial x^2}\} = k\mathcal{F}\{\frac{\partial^2 u}{\partial x^2}\} = -k\alpha^2 U$

$\therefore \frac{dU}{dt} = -k\alpha^2 U \Rightarrow U(\alpha,t) = c e^{-k\alpha^2 t}$

[Note : the Fourier transform changes our PDE into an ODE!]

Initial condition :

$$\mathcal{F}\{u(x,0)\} = \int_{-\infty}^{\infty} u(x,0) e^{i\alpha x} dx = \int_{-1}^{1} u_0 e^{i\alpha x} dx = \frac{u_0}{i\alpha}(e^{i\alpha} - e^{-i\alpha})$$

$$\therefore U(x,0) = \frac{u_0}{i\alpha}(e^{i\alpha} - e^{-i\alpha}) = \frac{u_0}{i\alpha}(2i \sin\alpha) = 2u_0 \frac{\sin\alpha}{\alpha}$$

Substitute the initial condition into $U(\alpha,t)$ :

$$Ce^{-k\alpha^2 \cdot 0} = 2u_0 \frac{\sin\alpha}{\alpha} \quad \Rightarrow \quad C = 2u_0 \frac{\sin\alpha}{\alpha}$$

$$\therefore \quad U(\alpha,t) = 2u_0 \frac{\sin\alpha}{\alpha} e^{-k\alpha^2 t}$$

Take the inverse Fourier transform :

$$u(x,t) = \frac{u_0}{\pi}\int_{-\infty}^{\infty} \frac{\sin\alpha}{\alpha} e^{-k\alpha^2 t} e^{-i\alpha x}\, d\alpha$$

$$= \frac{u_0}{\pi}\int_{-\infty}^{\infty} \frac{\sin\alpha}{\alpha} e^{-k\alpha^2 t}\left[\cos(\alpha x) - i\sin(\alpha x)\right] d\alpha$$

$$= \frac{u_0}{\pi}\int_{-\infty}^{\infty} \frac{\sin\alpha}{\alpha} e^{-k\alpha^2 t}\cos(\alpha x)\, d\alpha - \frac{u_0 i}{\pi}\int_{-\infty}^{\infty} \underbrace{\frac{\sin\alpha}{\alpha} e^{-k\alpha^2 t}\sin(\alpha x)\, d\alpha}_{\text{odd in } \alpha}$$

Hence $$u(x,t) = \frac{u_0}{\pi}\int_{-\infty}^{\infty} \frac{\sin\alpha \cos\alpha x}{\alpha} e^{-k\alpha^2 t}\, d\alpha .$$

Summary of the procedure :

$$\begin{tikzcd}
\boxed{\text{PDE in } u(x,t)} \arrow[d, "\text{F.T.}"'] & \boxed{u(x,t)} \\
\boxed{\text{ODE in } U(\alpha,t)} \arrow[r, "\text{solve}"] & \boxed{U(\alpha,t)} \arrow[u, "\text{I.F.T.}"']
\end{tikzcd}$$