TW364 Lecture 19

Fourier integral of f on $(-\infty, \infty)$:

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[A(\alpha)\cos(\alpha x) + B(\alpha)\sin(\alpha x)\right] d\alpha$$

where $A(\alpha) = \int_{-\infty}^{\infty} f(x)\cos(\alpha x)\,dx$, $B(\alpha) = \int_{-\infty}^{\infty} f(x)\sin(\alpha x)\,dx$.

For an even function f we get the cosine integral:

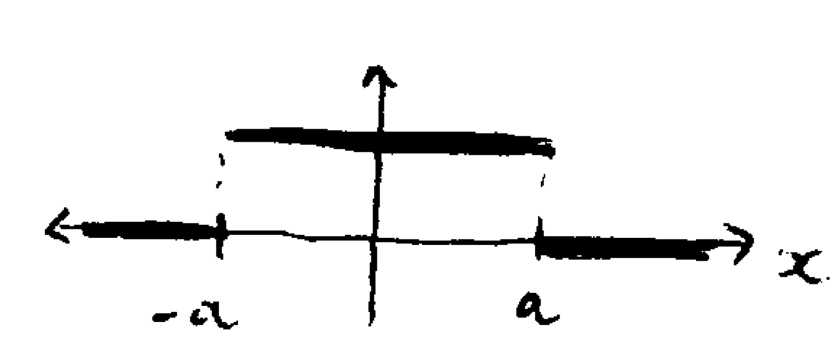
$$f(x) = \frac{1}{\pi}\int_0^\infty A(\alpha)\cos(\alpha x)\,d\alpha, \quad \text{where } A(\alpha) = 2\int_0^\infty f(x)\cos(\alpha x)\,dx$$

For an odd function f we get the sine integral:

$$f(x) = \frac{1}{\pi}\int_0^\infty B(\alpha)\sin(\alpha x)\,d\alpha, \quad \text{where } B(\alpha) = 2\int_0^\infty f(x)\sin(\alpha x)\,dx$$

Example

Find the Fourier integral of $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$



This function is even, so we determine the cosine integral:

$$A(\alpha) = 2\int_0^\infty f(x)\cos(\alpha x)\,dx = 2\int_0^a \cos(\alpha x)\,dx = \frac{2\sin(a\alpha)}{\alpha}$$

$$\therefore f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin(a\alpha)\cos(\alpha x)}{\alpha}\,d\alpha.$$

Half-line expansions

We can use cosine or sine integrals to expand a function defined on $(0, \infty)$ as either an even or odd function on $(-\infty, \infty)$.

Example

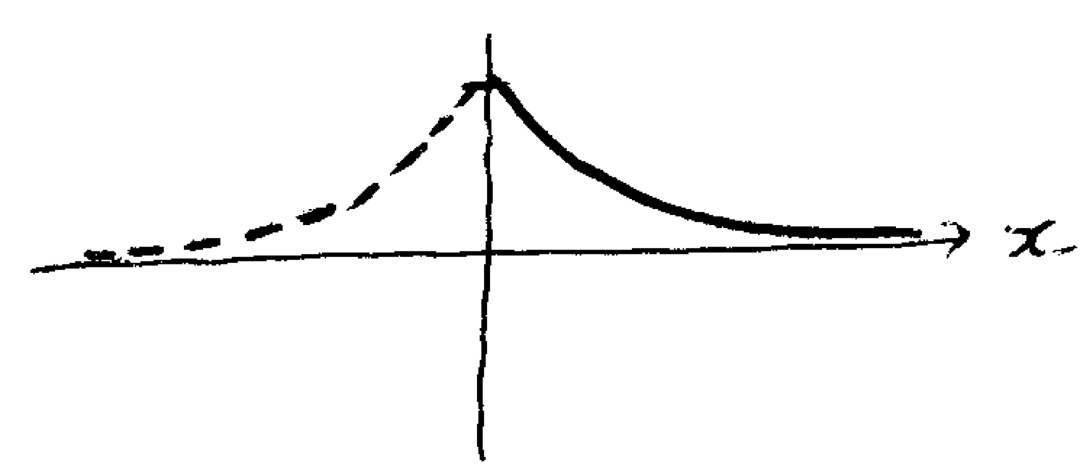
Represent $f(x) = e^{-x}, \quad x > 0$

(a) by a cosine integral

(b) by a sine integral

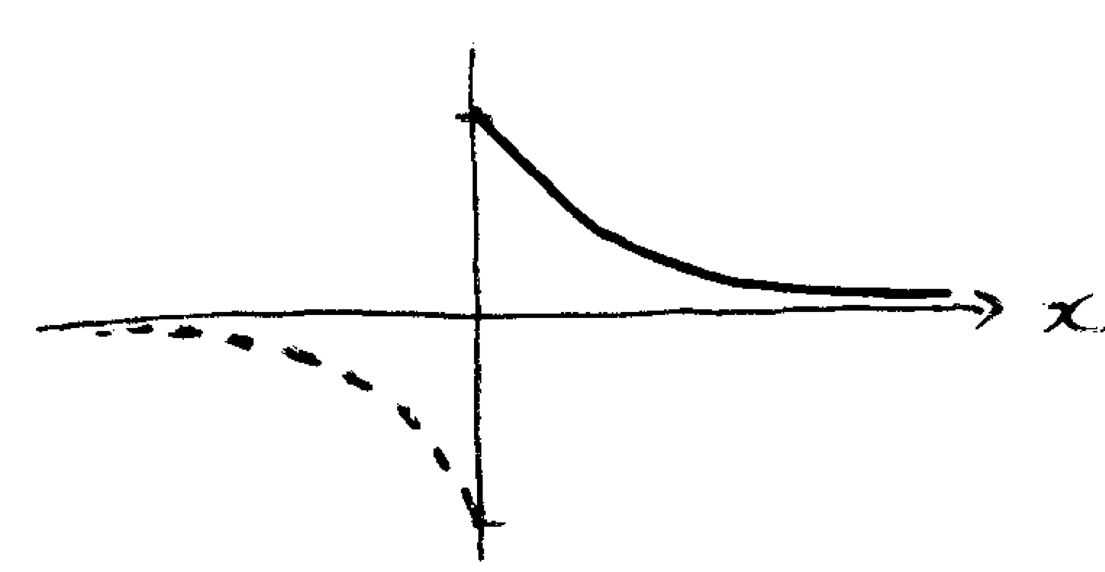
(a) $A(\alpha) = 2\int_0^\infty e^{-x} \cos(\alpha x)\, dx = \frac{2}{1+\alpha^2}$

$$\therefore f(x) = \frac{2}{\pi} \int_0^\infty \frac{\cos(\alpha x)}{1+\alpha^2}\, d\alpha$$



(b) $B(\alpha) = 2\int_0^\infty e^{-x} \sin(\alpha x)\, dx = \frac{2\alpha}{1+\alpha^2}$

$$\therefore f(x) = \frac{2}{\pi} \int_0^\infty \frac{\alpha \sin(\alpha x)}{1+\alpha^2}\, d\alpha$$



Partial integrals

We can examine the convergence of a Fourier integral, in a manner similar to plotting partial sum approximations of a Fourier series.

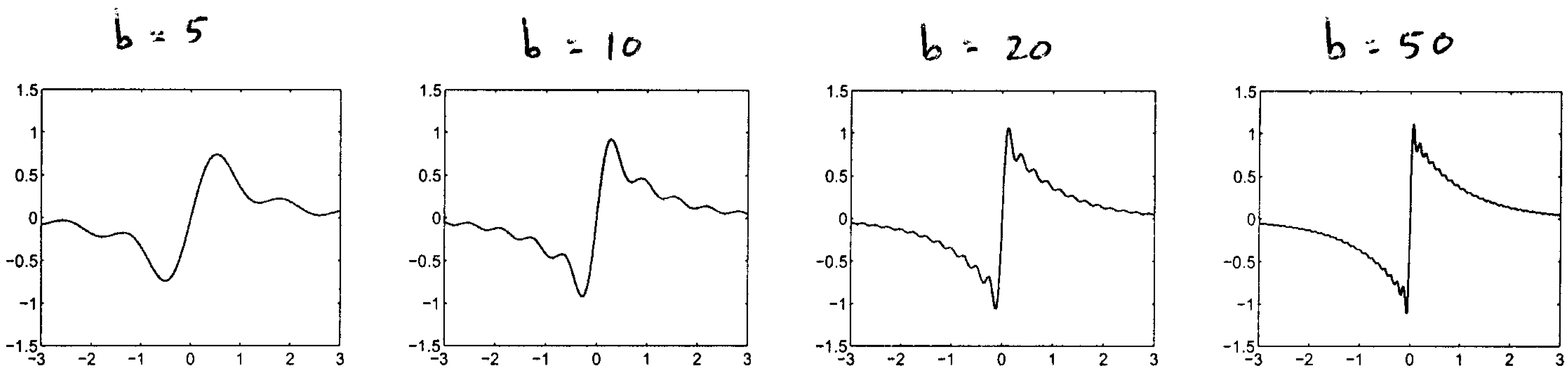
A Fourier integral is of the form

$$f(x) = \lim_{b \to \infty} \frac{1}{\pi} \int_0^b F(x, \alpha)\, d\alpha,$$

thus $f(x) \approx \frac{1}{\pi} \int_0^b F(x, \alpha)\, d\alpha$ for large b.

Partial integrals for part (b) of the previous example :

[Note : we would also typically use numerical integration to evaluate.]



Complex form

$$f(x) = \frac{1}{\pi} \int_0^\infty \left[\left(\int_{-\infty}^\infty f(t) \cos(\alpha t)\, dt \right) \cos(\alpha x) + \left(\int_{-\infty}^\infty f(t) \sin(\alpha t)\, dt \right) \sin(\alpha x) \right] d\alpha$$

$$= \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(t) \left[\cos(\alpha t) \cos(\alpha x) + \sin(\alpha t) \sin(\alpha x) \right] dt\, d\alpha$$

$$= \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(t) \cos \alpha(t - x)\, dt\, d\alpha$$

$$= \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty f(t) \cos \alpha(t - x)\, dt\, d\alpha$$

[the integrand is even in α]

Note : $i\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\sin\alpha(t-x)\,dt\,d\alpha = 0$ [this integrand is <u>odd</u> in α]

Therefore

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\left[\cos\alpha(t-x) + i\sin\alpha(t-x)\right]dt\,d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\,e^{i\alpha(t-x)}\,dt\,d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)\,e^{i\alpha t}\,dt\right)e^{-i\alpha x}\,d\alpha$$

<u>Fourier integral in complex form</u> :

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\,e^{-i\alpha x}\,d\alpha$$

with $F(\alpha) = \int_{-\infty}^{\infty} f(x)\,e^{i\alpha x}\,dx$.