TW364 Lecture 18

14.3 FOURIER INTEGRAL

We've used Fourier series to represent a function f defined on $[-p,p]$ or $[0,L]$. When f and f' are piecewise continuous on such an interval, a Fourier series converges to the periodic extension of f outside the interval.

In a sense, Fourier series are associated with periodic functions.

We now derive a means of representing certain kinds of nonperiodic functions defined on $(-\infty, \infty)$.

Fourier series to Fourier integral

Consider the Fourier series of a function f on $(-p,p)$:

$$f(x) = \frac{1}{2p}\int_{-p}^{p} f(t)\,dt \quad \ldots$$

$$+ \frac{1}{p}\sum_{n=1}^{\infty}\left[\left(\int_{-p}^{p} f(t)\cos\frac{n\pi t}{p}\,dt\right)\cos\frac{n\pi x}{p} + \left(\int_{-p}^{p} f(t)\sin\frac{n\pi t}{p}\,dt\right)\sin\frac{n\pi x}{p}\right]$$

Let $\alpha_n = \frac{n\pi}{p}$

$$\Delta\alpha = \alpha_{n+1} - \alpha_n = \frac{\pi}{p}$$

$$f(x) = \frac{1}{2\pi}\left(\int_{-p}^{p} f(t)\,dt\right)\Delta\alpha \quad \ldots \qquad \boxed{*}$$

$$+ \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\left(\int_{-p}^{p} f(t)\cos(\alpha_n t)\,dt\right)\cos(\alpha_n x) + \left(\int_{-p}^{p} f(t)\sin(\alpha_n t)\,dt\right)\sin(\alpha_n x)\right]\Delta\alpha$$

Let's expand the interval $(-p,p)$ by making $p \to \infty$.

Then $\Delta\alpha \to 0$

$\alpha_n \to \alpha$ (a continuous variable)

If $\int_{-\infty}^{\infty} f(t)\,dt$ exists, the limit of the first term in $\boxed{*}$ is zero (since $\Delta\alpha \to 0$).

The second term in $\boxed{*}$ has the form

$$\lim_{\Delta\alpha \to 0} \sum_{n=1}^{\infty} F(\alpha_n)\,\Delta\alpha = \int_0^{\infty} F(\alpha)\,d\alpha. \qquad [\text{"Riemann sum"}]$$

So we have the following:

<u>The Fourier integral of f on $(-\infty, \infty)$</u>

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left[A(\alpha)\cos(\alpha x) + B(\alpha)\sin(\alpha x)\right]d\alpha$$

with $$A(\alpha) = \int_{-\infty}^{\infty} f(x)\cos(\alpha x)\,dx$$

$$B(\alpha) = \int_{-\infty}^{\infty} f(x)\sin(\alpha x)\,dx$$

Conditions for convergence

Theorem 14.3.1, p. 521 :

If f and f' are piecewise continuous on every finite interval, and f is absolutely integrable on $(-\infty, \infty)$, the Fourier integral converges to $f(x)$ for all point continuities.

At a point discontinuity x, the Fourier integral converges to the average of the right- and left-hand limits of f at x, that is $\frac{1}{2}\left(f(x^+) + f(x^-)\right)$.

Example

Find the Fourier integral of $f(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 < x < 2 \\ 0, & x > 2 \end{cases}$

$$A(\alpha) = \int_{-\infty}^{\infty} f(x) \cos(\alpha x)\, dx = \int_0^2 \cos(\alpha x)\, dx = \frac{\sin(2\alpha)}{\alpha}$$

$$B(\alpha) = \int_{-\infty}^{\infty} f(x) \sin(\alpha x)\, dx = \int_0^2 \sin(\alpha x)\, dx = \frac{1 - \cos(2\alpha)}{\alpha}$$

$$\therefore f(x) = \frac{1}{\pi} \int_0^{\infty} \left[\frac{\sin(2\alpha)}{\alpha} \cos(\alpha x) + \frac{1-\cos(2\alpha)}{\alpha} \sin(\alpha x) \right] d\alpha$$

$\vdots$ trig identities...

$$= \frac{2}{\pi} \int_0^{\infty} \frac{\sin\alpha \cos\alpha(x-1)}{\alpha}\, d\alpha .$$