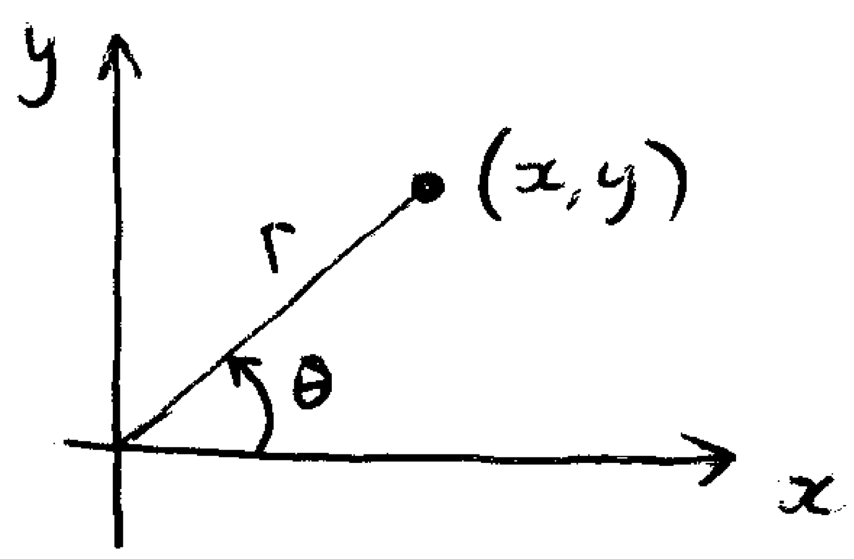
TW364 Lecture 17

13.1 POLAR COORDINATES



$x = r\cos\theta$ and $y = r\sin\theta$

$r^2 = x^2 + y^2$ and $\tan\theta = \dfrac{y}{x}$

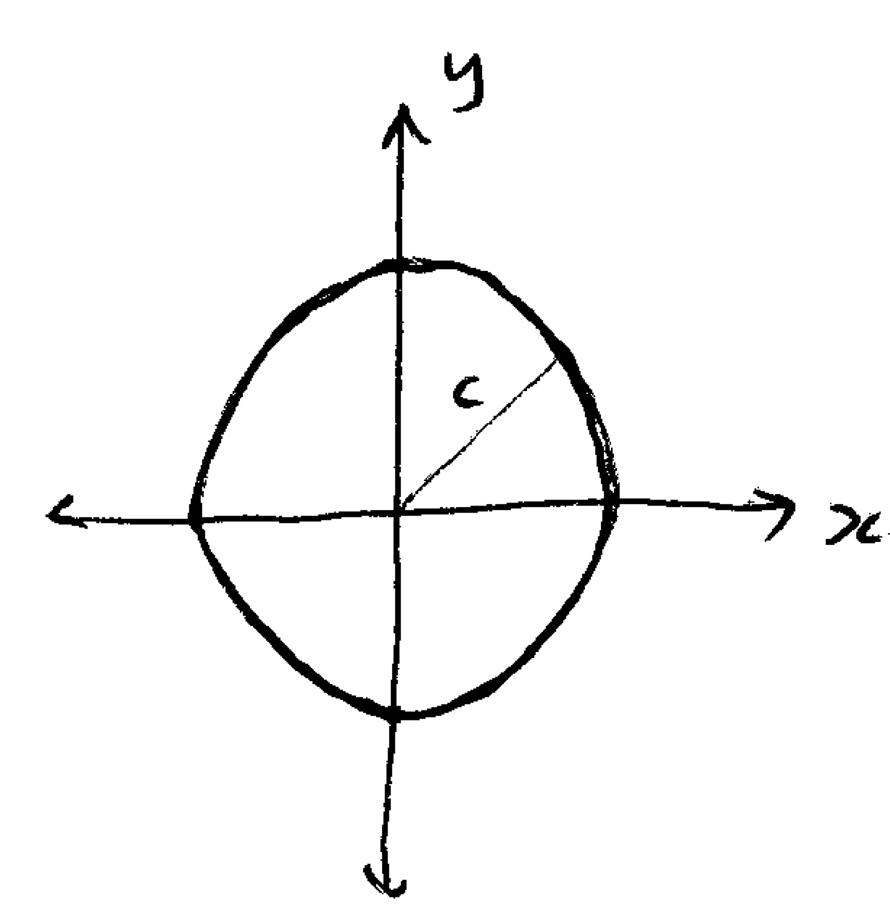
For some problems involving Laplace's equation, it might be more convenient to work in polar coordinates.

It can be shown (APPENDIX A) that if $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$,

then $$\boxed{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0}$$

Example

Find the steady-state temperature $u(r,\theta)$ in a circular disk of radius c, if the boundary's temperature is fixed at $f(\theta)$.



$\therefore$ Solve $$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

subject to $u(c,\theta) = f(\theta)$, $0 \le \theta \le 2\pi$.

We note an additional <u>implicit</u> condition:

$u(r,\theta)$ must be periodic in θ, with period 2π:

$$u(r, \theta + 2\pi) = u(r,\theta).$$

Assume $u(r,\theta) = R(r)\,\Theta(\theta)$. The PDE becomes

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0 \Rightarrow \frac{\Theta''}{\Theta} = -\frac{r^2R'' + rR'}{R} = -\lambda$$

$$\therefore \quad \Theta'' + \lambda\Theta = 0$$

$$r^2R'' + rR' - \lambda R = 0$$

$u(r,\theta)$ must be periodic in θ, so $\Theta(\theta)$ must be periodic.

We seek nontrivial solutions to:

$$\Theta'' + \lambda\Theta = 0, \quad \Theta(\theta + 2\pi) = \Theta(\theta).$$

$$\lambda = 0 \quad : \quad \Theta(\theta) = c_1\theta + c_2 \qquad (1)$$

$$\lambda = -\alpha^2 : \quad \Theta(\theta) = c_1\cosh(\alpha\theta) + c_2\sinh(\alpha\theta) \qquad (2)$$

$$\lambda = \alpha^2 : \quad \Theta(\theta) = c_1\cos(\alpha\theta) + c_2\sin(\alpha\theta) \qquad (3)$$

We can dismiss (2) as inherently nonperiodic (unless $c_1 = c_2 = 0$).

Solution (1) is periodic if $c_1 = 0$ $\quad\Rightarrow \lambda_0 = 0$

Solution (3) is 2π-periodic if α is integer $\Rightarrow \lambda_n = n^2, \quad n = 1, 2, \ldots$

$$\therefore \quad \Theta_0 = c_1 \quad \text{and} \quad \Theta_n = c_1\cos(n\theta) + c_2\sin(n\theta), \quad n = 1, 2, \ldots$$

Next we seek a solution to :

$$r^2 R'' + r R' - \lambda R = 0$$

This is a so-called Cauchy-Euler equation (APPENDIX B)

with auxiliary equation $m^2 - \lambda = 0 \Rightarrow m = \pm\sqrt{\lambda}$.

$\lambda_0 = 0$: $m = 0$ (repeated real root)

$$R_0 = c_3 + c_4 \ln r$$

$\lambda_n = n^2$: $m_1 = n$, $m_2 = -n$ (two distinct real roots)

$$R_n = c_3 r^n + c_4 r^{-n}, \quad n = 1, 2, \ldots$$

We want $u(r,\theta)$ to be continuous and bounded everywhere in the disk. The only way for R_0 and R_n to be bounded at $r=0$ (disk centre) is for c_4 to be zero in both.

$\therefore$ $R_0 = c_3$ and $R_n = c_3 r^n$, $n = 1, 2, \ldots$

Now form $u_n = R_n \Theta_n$ and super-position :

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n \left[A_n \cos(n\theta) + B_n \sin(n\theta) \right].$$

By enforcing the remaining condition $u(c,\theta) = f(\theta)$, we have

$$f(\theta) = A_0 + \sum_{n=1}^{\infty} c^n \left[A_n \cos(n\theta) + B_n \sin(n\theta) \right],$$

which is the full Fourier series of $f(\theta)$. So.....

$$A_0 = \frac{a_0}{2} = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta$$

$$A_n = \frac{a_n}{c^n} = \frac{1}{c^n\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta$$

$$B_n = \frac{b_n}{c^n} = \frac{1}{c^n\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta$$

Note :

Since the integrands are 2π-periodic, we can integrate from 0 to 2π, instead of $-\pi$ to π.

Example

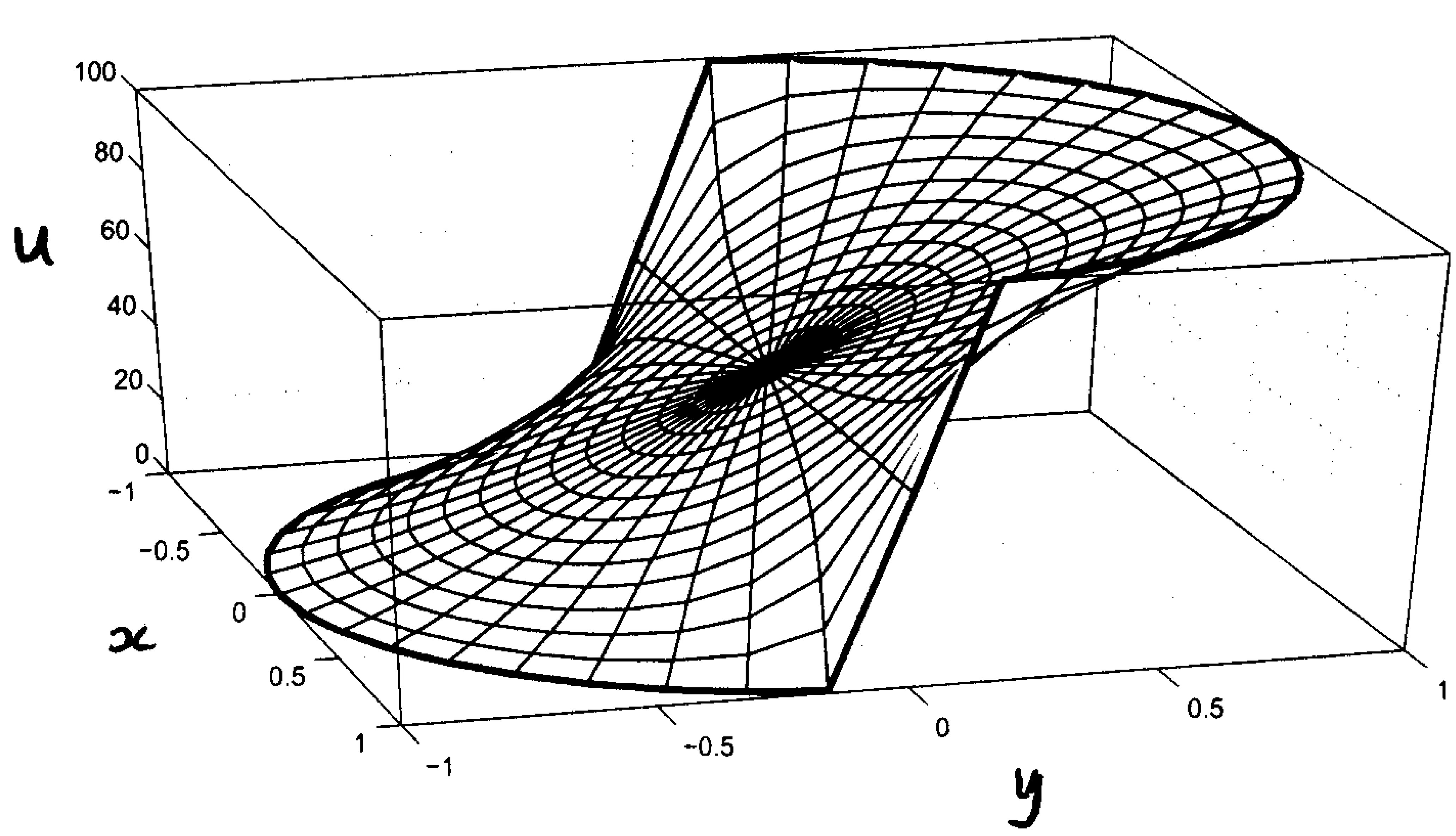
Suppose $c = 1$ and $u(1,\theta) = \begin{cases} 100, & 0 < \theta < \pi \\ 0, & \pi < \theta < 2\pi \end{cases}$

$$A_0 = \frac{1}{2\pi}\int_0^{\pi} 100\,d\theta = 50$$

$$A_n = \frac{1}{\pi}\int_0^{\pi} 100\cos(n\theta)\,d\theta = 0$$

$$B_n = \frac{1}{\pi}\int_0^{\pi} 100\sin(n\theta)\,d\theta = \frac{100}{\pi}\left(\frac{1-(-1)^n}{n}\right)$$

$$\therefore\; u(r,\theta) = 50 + \frac{100}{\pi}\sum_{n=1}^{\infty} r^n\left(\frac{1-(-1)^n}{n}\right)\sin(n\theta).$$



APPENDIX A

Cauchy-Euler equations are DE^s of the form:

$$ax^2 y'' + bxy' + cx = 0$$

To solve them, we substitute $y = x^m$, and solve for m:

$$ax^2 m(m-1) x^{m-2} + bxmx^{m-1} + cx^m = 0$$

$$am(m-1)x^m + bmx^m + cx^m = 0$$

$$\therefore \; am^2 + (b-a)m + c = 0$$

This is our auxiliary equation, from which we get m.

① Two distinct real roots (m_1 and m_2):

$$y = c_1 x^{m_1} + c_2 x^{m_2}$$

② Repeated real root (m):

$$y = c_1 x^m + c_2 x^m \ln x$$

③ Conjugate complex roots ($\alpha \pm i\beta$):

$$y = x^\alpha \left[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x) \right]$$

Details: pp. 164-165 in the textbook.

APPENDIX B

Laplace's equation : $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

Now ...

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}} \qquad \boxed{\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}}$$

$$\frac{\partial}{\partial x}(r^2) = \frac{\partial}{\partial x}(x^2 + y^2)$$

$$2r\frac{\partial r}{\partial x} = 2r\cos\theta$$

$$\therefore \frac{\partial r}{\partial x} = \cos\theta$$

$$\frac{\partial}{\partial x}(\tan\theta) = \frac{\partial}{r}\left(\frac{y}{x}\right)$$

$$(1+\tan^2\theta)\frac{\partial \theta}{\partial x} = -\frac{y}{x^2}$$

$$(x^2+y^2)\frac{\partial \theta}{\partial x} = -y$$

$$\therefore \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}$$

$$\frac{\partial}{\partial y}(r^2) = \frac{\partial}{\partial y}(x^2 + y^2)$$

$$2r\frac{\partial r}{\partial y} = 2r\sin\theta$$

$$\therefore \frac{\partial r}{\partial y} = \sin\theta$$

$$\frac{\partial}{\partial y}(\tan\theta) = \frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

$$(1+\tan^2\theta)\frac{\partial \theta}{\partial y} = \frac{1}{x}$$

$$(x^2+y^2)\frac{\partial \theta}{\partial y} = x$$

$$\therefore \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$$

Therefore $\dfrac{\partial u}{\partial x} = \cos\theta\dfrac{\partial u}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial u}{\partial \theta}$

$$\frac{\partial u}{\partial y} = \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \cos\theta\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \cos\theta \left[\cos\theta \frac{\partial^2 u}{\partial r^2} - \frac{\sin\theta}{r} \frac{\partial^2 u}{\partial r \partial \theta} + \frac{\sin\theta}{r^2} \frac{\partial u}{\partial \theta} \right]$$

$$- \frac{\sin\theta}{r} \left[\cos\theta \frac{\partial^2 u}{\partial r \partial u} - \sin\theta \frac{\partial u}{\partial r} - \frac{\sin\theta}{r} \frac{\partial^2 u}{\partial r^2} - \frac{\cos\theta}{r} \frac{\partial u}{\partial \theta} \right]$$

$$= \cos^2\theta \frac{\partial^2 u}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r} \frac{\partial^2 u}{\partial r \partial \theta} + \frac{\sin^2\theta}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\sin^2\theta}{r} \frac{\partial u}{\partial r} + \frac{2\sin\theta\cos\theta}{r^2} \frac{\partial u}{\partial \theta}.$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \sin\theta \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial y}\right) + \frac{\cos\theta}{r} \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial y}\right)$$

$$= \sin\theta \left[\sin\theta \frac{\partial^2 u}{\partial r^2} + \frac{\cos\theta}{r} \frac{\partial^2 u}{\partial r \partial \theta} - \frac{\cos\theta}{r^2} \frac{\partial u}{\partial \theta} \right]$$

$$+ \frac{\cos\theta}{r} \left[\sin\theta \frac{\partial^2 u}{\partial r \partial \theta} + \cos\theta \frac{\partial u}{\partial r} + \frac{\cos\theta}{r} \frac{\partial^2 u}{\partial \theta^2} - \frac{\sin\theta}{r} \frac{\partial u}{\partial \theta} \right]$$

$$= \sin^2\theta \frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r} \frac{\partial^2 u}{\partial r \partial \theta} + \frac{\cos^2\theta}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cos^2\theta}{r} \frac{\partial u}{\partial r} - \frac{2\sin\theta\cos\theta}{r^2} \frac{\partial u}{\partial \theta}.$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}.$$

Laplace's equation in polar coordinates:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0.$$