TW364 Lecture 16

12.6 NONHOMOGENEOUS BVPs

Nonhomogeneous BVPs (where the PDE and/or boundary conditions are nonhomogeneous) are, in general, not readily solved by means of separation of variables.

However, it might be possible to employ a change of variables that transforms the nonhomogeneous BVP into two problems:

* a relatively simple BVP with an ODE,
* and a homogeneous BVP with a PDE.

Note: this is "Method 1" in the textbook. There is also a "Method 2" that we'll skip for now.

Example

Solve the heat equation: $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$, $0 < x < 1$, $t \geq 0$

subject to

$u(0,t) = 0$

$u(1,t) = 100$ ← nonhomogeneous boundary condition!

$u(x,0) = 0$

The trick is: we define a new variable v such that

$u(x,t) = v(x,t) + \Psi(x)$, where Ψ is the steady-state solution.

Since Ψ is the steady-state:

$$\frac{\partial \Psi}{\partial t} = 0 \Rightarrow \frac{\partial^2 \Psi}{\partial x^2} = 0 \Rightarrow \Psi(x) = C_1 x + C_2$$

The steady-state must satisfy the boundary conditions:

$$\Psi(0) = 0 \quad : \quad C_1 \cdot 0 + C_2 = 0 \Rightarrow C_2 = 0$$

$$\Psi(1) = 100 \quad : \quad C_1 \cdot 1 + C_2 = 100 \Rightarrow C_1 = 100$$

$\therefore \Psi(x) = 100x$

Therefore $v(x,t) = u(x,t) - 100x$.

Note that $\frac{\partial v}{\partial t} = \frac{\partial u}{\partial t}$ and $k\frac{\partial^2 v}{\partial x^2} = k\frac{\partial^2 u}{\partial x^2}$

Also, $u(0,t) = 0 \Rightarrow v(0,t) + \Psi(0) = 0 \Rightarrow v(0,t) = 0$

$u(1,t) = 100 \Rightarrow v(1,t) + \Psi(1) = 100 \Rightarrow v(1,t) = 0$

$u(x,0) = 0 \Rightarrow v(x,0) + \Psi(x) = 0 \Rightarrow v(x,0) = -100x$

So we must solve $\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2}$

subject to $\underbrace{v(0,t) = 0, \ v(1,t) = 0}_{\text{homogeneous boundary conditions!}}$, $v(x,0) = -100x$

Using the result in Lecture 11, with $L=1$, $f(x) = -100x$,

$$v(x,t) = \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-kn^2\pi^2 t} \sin(n\pi x).$$

A solution to the original problem is therefore

$$u(x,t) = 100x + \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-kn^2\pi^2 t} \sin(n\pi x).$$

Wave equation with gravity

We can use the following nonhomogeneous BVP to model the vibration of a string with the effects of gravity taken into account:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} - g$$

This term makes the PDE nonhomogeneous.

subject to

$$u(0,t) = 0, \quad u(1,t) = 0$$
$$u(x,0) = 0, \quad u_t(x,0) = 0$$

Here we suppose that the string falls from rest, from the horizontal position.

Again, we write $u(x,t) = v(x,t) + \psi(x)$, where $v(x,t)$ will be a solution to the homogeneous problem.

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 v}{\partial t^2} + \frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 v}{\partial t^2}$$

$$a^2 \frac{\partial^2 u}{\partial x^2} - g = a^2 \frac{\partial^2 v}{\partial x^2} + a^2 \frac{\partial^2 \psi}{\partial x^2} - g$$

For v to be solved from a homogeneous BVP, we must insist

$$a^2 \frac{\partial^2 \psi}{\partial x^2} - g = 0$$

We also want ψ to satisfy the given boundary conditions, so that v can satisfy homogeneous boundary conditions.

So we must solve $a^2 \frac{\partial^2 \psi}{\partial x^2} - g = 0$, $\psi(0) = 0$, $\psi(1) = 0$.

$$\psi'' = \frac{g}{a^2} \Rightarrow \psi' = \frac{g}{a^2} x + c_1 \Rightarrow \psi = \frac{g}{2a^2} x^2 + c_1 x + c_2$$

$$\psi(0) = 0 : \frac{g}{2a^2}(0)^2 + c_1 \cdot 0 + c_2 = 0 \Rightarrow c_2 = 0$$

$$\psi(1) = 0 : \frac{g}{2a^2}(1)^2 + c_1 \cdot 1 + c_2 = 0 \Rightarrow c_1 = -\frac{g}{2a^2}$$

$$\therefore \psi(x) = \frac{g}{2a^2}(x^2 - x).$$

To get v we must solve $\frac{\partial^2 v}{\partial t^2} = a^2 \frac{\partial^2 v}{\partial x^2}$,

subject to:

$$u(0,t) = 0 \Rightarrow v(0,t) + \psi(0) = 0 \Rightarrow v(0,t) = 0$$

$$u(1,t) = 0 \Rightarrow v(1,t) + \psi(1) = 0 \Rightarrow v(1,t) = 0$$

$$u(x,0) = 0 \Rightarrow v(x,0) + \psi(x) = 0 \Rightarrow v(x,0) = -\psi(x)$$

$$u_t(x,0) = 0 \Rightarrow v_t(x,0) + \psi_t(x) = 0 \Rightarrow v_t(x,0) = 0$$

Using the result in Lecture 13,

$$v(x,t) = \frac{2g}{a^2\pi^3} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \cos(an\pi t) \sin(n\pi x).$$

A solution to the original problem is therefore

$$u(x,t) = \frac{g}{2a^2}(x^2 - x) + \frac{2g}{a^2\pi^3} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^3} \cos(an\pi t) \sin(n\pi x).$$