TW364 Lecture 15

Dirichlet problems

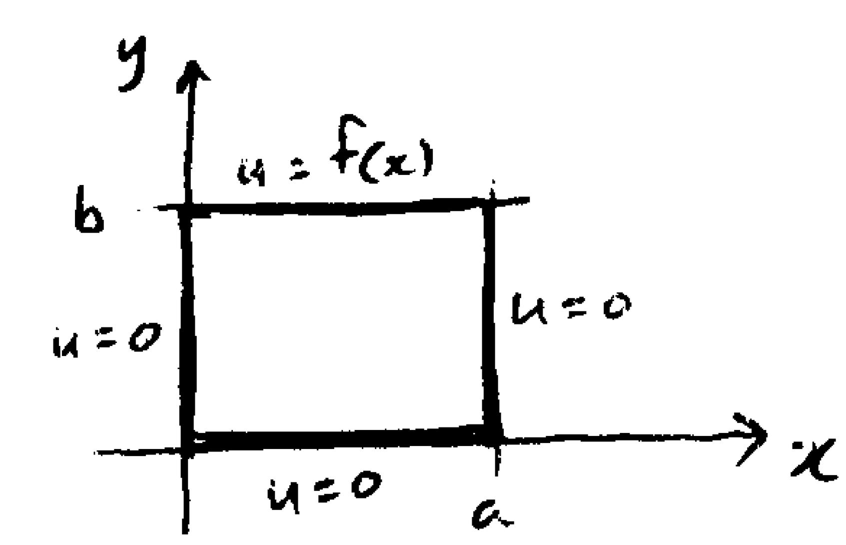
A boundary value problem consisting of an <u>elliptic</u> PDE, for which the <u>function values</u> (not derivatives) are specified on the entire boundary of a bounded region, is called a Dirichlet problem.

Example

Solve $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ [Laplace's eqn.]

subject to $u(0,y) = 0, \quad u(a,y) = 0$

$u(x,0) = 0, \quad u(x,b) = f(x)$



Assume $u = XY$, separate variables ...

$$X'' + \lambda X = 0 \quad \text{and} \quad Y'' - \lambda Y = 0$$

(*) The first of these ODEs, together with $u(0,y)=0$ and $u(a,y)=0$, give the Sturm-Liouville problem:

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X(a) = 0.$$

$$\Rightarrow \lambda_n = \frac{n^2\pi^2}{a^2} \quad \text{and} \quad X_n = \sin\frac{n\pi x}{a}, \quad n = 1, 2, \ldots$$

Next we solve $Y_n'' - \lambda_n Y_n = 0$ with $Y_n(0) = 0$ ← from $u(x,0) = 0$

$$Y_n = c_2 \sinh \frac{n\pi y}{a}$$

We use the superposition principle to form

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi y}{a} \sin \frac{n\pi x}{a}.$$

Enforcing the remaining condition $u(x,b) = f(x)$ leads to

$f(x) = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi x}{a}$, which is the sine series of $f(x)$.

Hence $A_n \sinh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a}\, dx$

$\therefore$ Our solution to this Dirichlet problem is

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi y}{a} \sin \frac{n\pi x}{a},$$

$$\text{with } A_n = \frac{2}{a \sinh \frac{n\pi b}{a}} \int_0^a f(x) \sin \frac{n\pi x}{a}\, dx.$$

Note:

A Dirichlet problem on a rectangular region can be readily solved by separation of variables when homogeneous boundary conditions are specified on two parallel boundaries (because that leads to a Sturm-Liouville problem; see (*) in the example above).

But what about a problem like this:

$$\text{solve} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

subject to $u(0,y) = F(y)$, $u(a,y) = G(y)$
$u(x,0) = f(x)$, $u(x,b) = g(x)$ ....?

The method of separation of variables won't lead us to a solution, but it turns out that we can break the problem into two problems:

<u>Problem 1</u>: Solve $\dfrac{\partial^2 u_1}{\partial x^2} + \dfrac{\partial^2 u_1}{\partial y^2} = 0$

subject to $u_1(0,y) = 0$, $u_1(a,y) = 0$
$u_1(x,0) = f(x)$, $u_1(x,b) = g(x)$

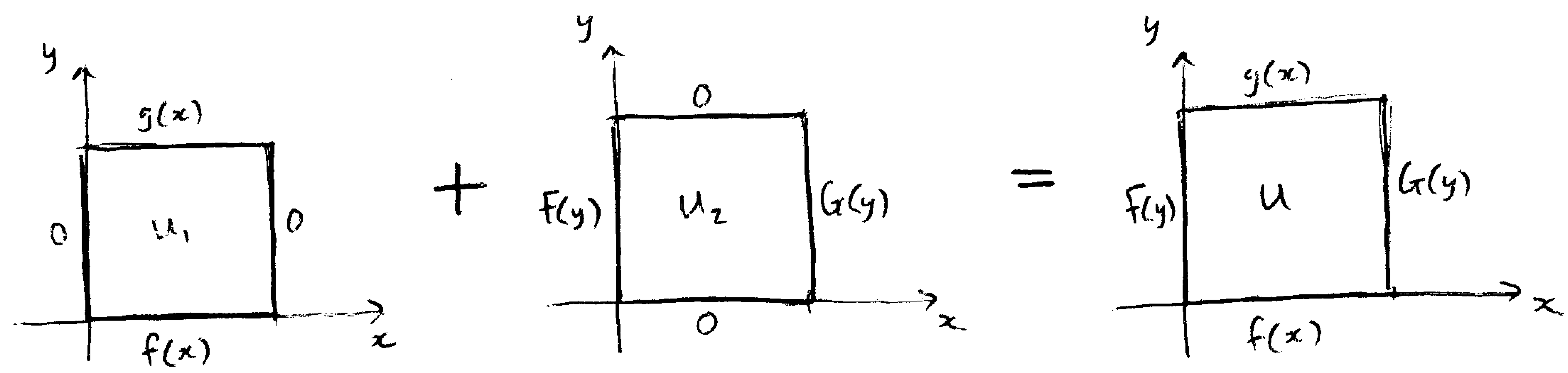
<u>Problem 2</u>: Solve $\dfrac{\partial^2 u_2}{\partial x^2} + \dfrac{\partial^2 u_2}{\partial y^2} = 0$

subject to $u_2(0,y) = F(y)$, $u_2(a,y) = G(y)$
$u_2(x,0) = 0$, $u_2(x,b) = 0$

We solve these problems (sep. of variables will work!), and define

$$u(x,y) = u_1(x,y) + u_2(x,y).$$

It is easily verified that this $u(x,y)$ is a solution to the original boundary value problem.



I'll leave it as an exercise for you to show that

$$u_1(x,y) = \sum_{n=1}^{\infty}\left[A_n \cosh\frac{n\pi y}{a} + B_n \sinh\frac{n\pi y}{a}\right]\sin\frac{n\pi x}{a},$$

with

$$A_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx\ ;\quad B_n = \frac{1}{\sinh\frac{n\pi b}{a}}\left[\frac{2}{a}\int_0^a g(x)\sin\frac{n\pi x}{a}\,dx - A_n\cosh\frac{n\pi b}{a}\right],$$

$$u_2(x,y) = \sum_{n=1}^{\infty}\left[C_n \cosh\frac{n\pi x}{b} + D_n \sinh\frac{n\pi x}{b}\right]\sin\frac{n\pi y}{b},$$

with

$$C_n = \frac{2}{b}\int_0^b F(y)\sin\frac{n\pi y}{b}\,dy\ ;\quad D_n = \frac{1}{\sinh\frac{n\pi a}{b}}\left[\frac{2}{b}\int_0^b G(y)\sin\frac{n\pi y}{b}\,dy - C_n\cosh\frac{n\pi a}{b}\right].$$

The extremum principle

A solution $u(x,y)$ of Laplace's equation within a bounded region R with boundary B takes on its maximum and minimum values <u>on</u> B.

In addition, it can be proved that $u(x,y)$ can have <u>no</u> local minima or maxima in the <u>interior</u> of R.