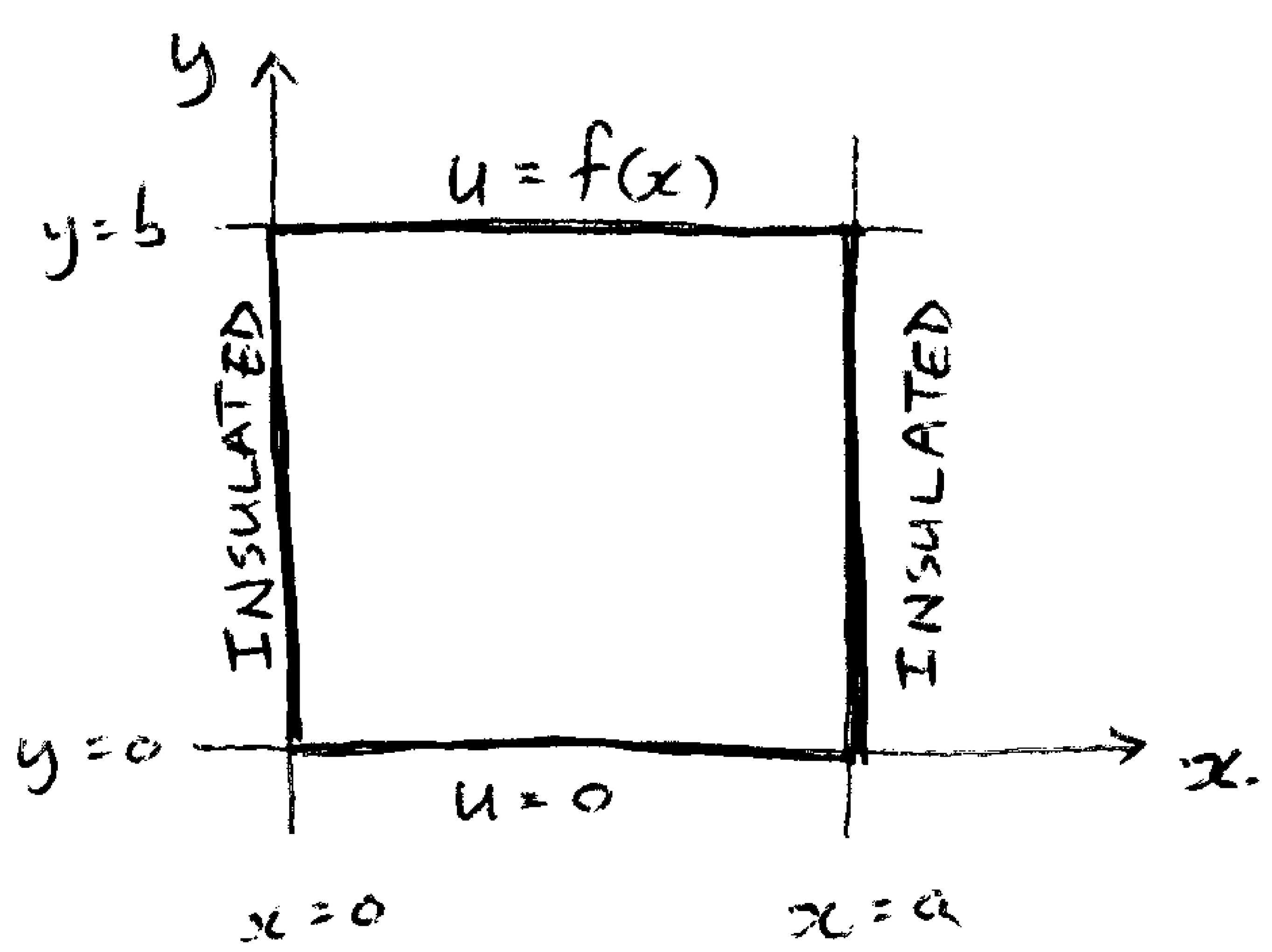
# 12.5 LAPLACE'S EQUATION

Suppose we wish to find the steady-state temperature $u(x,y)$ at every point $(x,y)$ in a rectangular plate, under the following boundary conditions:




We must solve the following boundary value problem:

Laplace's equation: $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Boundary conditions:
$$u_x(0,y) = 0$$
$$u_x(a,y) = 0$$
$$u(x,0) = 0$$
$$u(x,b) = f(x)$$

Assume $u(x,y) = X(x)\,Y(y)$ :

$$X''Y + XY'' = 0 \quad \Rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

$$\therefore \quad X'' + \lambda X = 0 \quad \text{and} \quad Y'' - \lambda Y = 0$$

The three homogeneous boundary conditions give :

$$u_x(0,y) = 0 \Rightarrow X'(0)\,Y(y) = 0 \Rightarrow X'(0) = 0$$

$$u_x(a,y) = 0 \Rightarrow X'(a)\,Y(y) = 0 \Rightarrow X'(a) = 0$$

$$u(x,0) = 0 \Rightarrow X(x)\,Y(0) = 0 \Rightarrow Y(0) = 0$$

In Tutorial 2, Problem 5, we solved the Sturm-Liouville problem :

$$X'' + \lambda X = 0, \quad X'(0) = 0, \quad X'(a) = 0.$$

$$\Rightarrow \quad \lambda_n = \frac{n^2\pi^2}{a^2}, \quad n = 0,1,2,\ldots \qquad [\text{Note} : \lambda_0 = 0 \text{ and } \lambda_1, \lambda_2, \ldots > 0]$$

$$X_n = \cos\frac{n\pi x}{a}, \quad n = 0,1,2,\ldots \qquad [\text{Note} : X_0 = 1]$$

Next we consider $Y'' - \lambda Y = 0$, $Y(0) = 0$ :

For $\lambda_0 = 0$ : $Y_0'' = 0 \Rightarrow Y_0 = c_1 y + c_2$

$Y_0(0) = 0$ : $0 = c_1 \cdot 0 + c_2 \Rightarrow c_2 = 0$

$\therefore Y_0 = c_1 y$

For $\lambda_n = \frac{n^2\pi^2}{a^2}$, $n = 1, 2, \ldots$ : $\quad Y_n'' = \frac{n^2\pi^2}{a^2} Y_n$

$$\Rightarrow Y_n = C_3 \cosh \frac{n\pi y}{a} + C_4 \sinh \frac{n\pi y}{a}$$

$$Y_n(0) = 0 : \quad 0 = C_3 \cdot 1 + C_4 \cdot 0 \Rightarrow C_3 = 0$$

$$\therefore Y_n = C_4 \sinh \frac{n\pi y}{a}$$

Using the fact that $u = XY$, and the superposition principle, we form

$$u(x,y) = A_0 y + \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi y}{a} \cos \frac{n\pi x}{a}.$$

Now we enforce the last boundary condition, $u(x,b) = f(x)$, to find the constants $A_0, A_1, A_2, \ldots$ :

$$f(x) = A_0 b + \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi b}{a} \cos \frac{n\pi x}{a}.$$

We recognize this as the cosine series of $f(x)$, so

$$A_0 b = \frac{a_0}{2} = \frac{1}{a}\int_0^a f(x)\,dx \Rightarrow A_0 = \frac{1}{ab}\int_0^a f(x)\,dx$$

and $$A_n \sinh \frac{n\pi b}{a} = a_n = \frac{2}{a}\int_0^a f(x) \cos \frac{n\pi x}{a}\,dx$$

$$\Rightarrow A_n = \frac{2}{a \sinh \frac{n\pi b}{a}} \int_0^a f(x) \cos \frac{n\pi x}{a}\,dx.$$

To summarize, a solution to our BVP is :

$$u(x,y) = A_0 y + \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi y}{a} \cos \frac{n\pi x}{a}$$

with $$A_0 = \frac{1}{ab}\int_0^a f(x)\,dx$$

$$A_n = \frac{2}{a\sinh \frac{n\pi b}{a}} \int_0^a f(x) \cos \frac{n\pi x}{a}\,dx \,, \quad n = 1, 2, \cdots$$

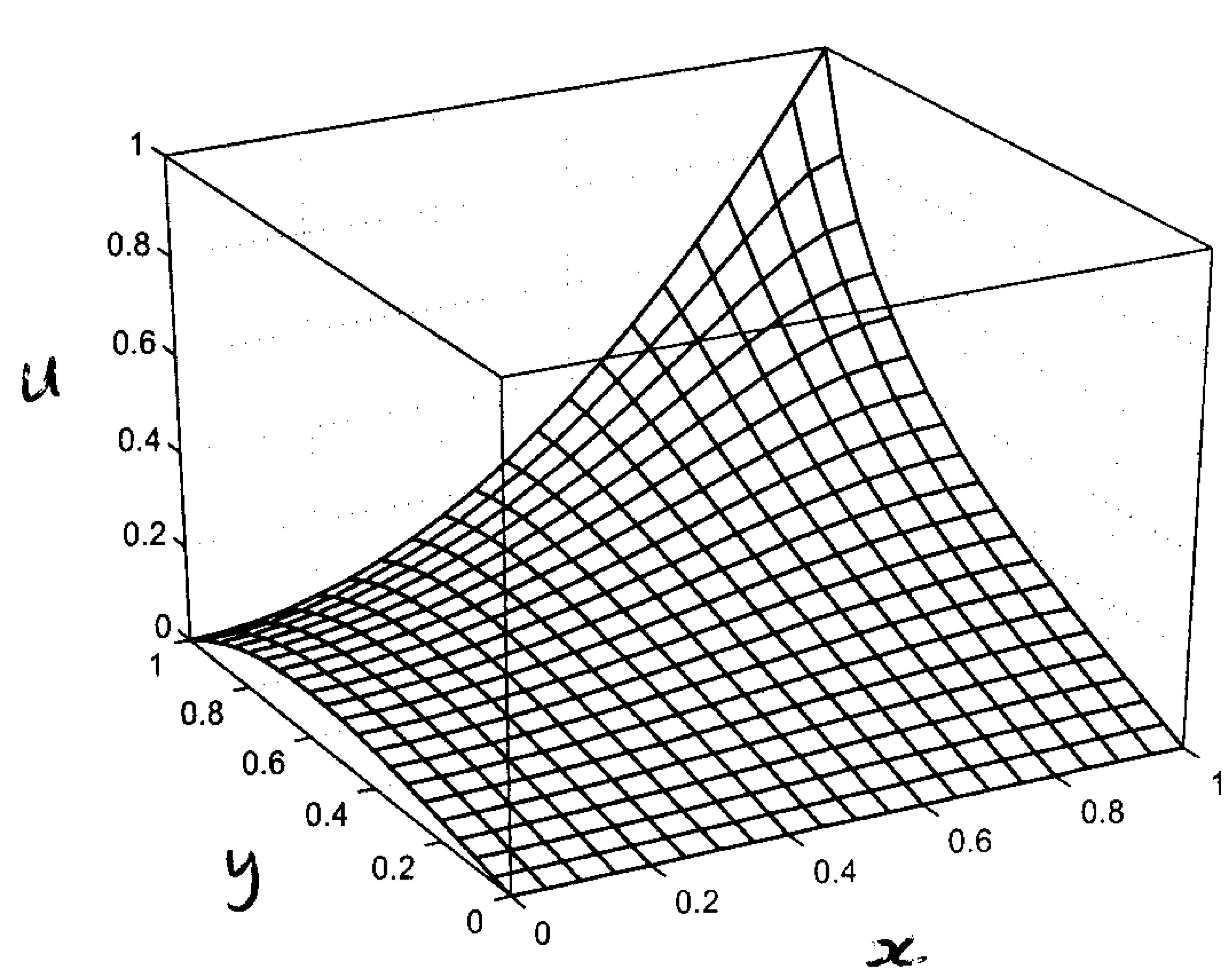
## Example

Suppose $a=1$, $b=1$, $f(x) = x^2$.

Then $$A_0 = \int_0^1 x^2\,dx = \frac{1}{3}$$

$$A_n = \frac{2}{\sinh(n\pi)} \int_0^1 x^2 \cos(n\pi x)\,dx = \frac{4(-1)^n}{\pi^2 n^2 \sinh(\pi n)}$$

$$\therefore \; u(x,y) = \frac{1}{3} y + \sum_{n=1}^{\infty} \frac{4(-1)^n}{\pi^2 n^2 \sinh(\pi n)} \sinh(n\pi y) \cos(n\pi x) .$$

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