TW364 Lecture 13

In the previous lecture we solved the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \quad ; \; u(0,t) = 0 ; \; u(L,t) = 0 ; \; u(x,0) = f(x) ; \; u_t(x,0) = g(x) .$$

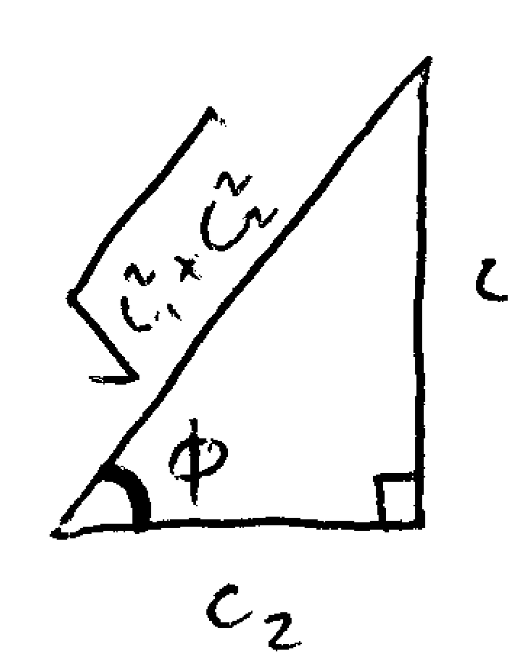
$$\Rightarrow \quad u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos \frac{an\pi t}{L} + B_n \sin \frac{an\pi t}{L} \right] \sin \frac{n\pi x}{L} ;$$

$$\text{with } A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx \quad \text{and} \quad B_n = \frac{2}{n\pi a} \int_0^L g(x) \sin \frac{n\pi x}{L} \, dx .$$

Amplitude-phase form (from AM244)

Any linear combination of a cosine- and sine-wave with the same frequency (say $\frac{\gamma}{2\pi}$) can be expressed as a single sine-wave ("simple harmonic motion"):

$$
\begin{aligned}
& c_1 \cos \gamma t + c_2 \sin \gamma t \\
&= \sqrt{c_1^2 + c_2^2} \left[\frac{c_1}{\sqrt{c_1^2 + c_2^2}} \cos \gamma t + \frac{c_2}{\sqrt{c_1^2 + c_2^2}} \sin \gamma t \right] \\
&= \sqrt{c_1^2 + c_2^2} \left[\sin\phi \cos \gamma t + \cos\phi \sin \gamma t \right] \\
&= \sqrt{c_1^2 + c_2^2} \sin(\gamma t + \phi) .
\end{aligned}
$$



Note: we define ϕ to be the angle for which

$$\sin\phi = c_1 / \sqrt{c_1^2 + c_2^2} \quad \text{and} \quad \cos\phi = c_2 / \sqrt{c_1^2 + c_2^2}$$

Standing waves

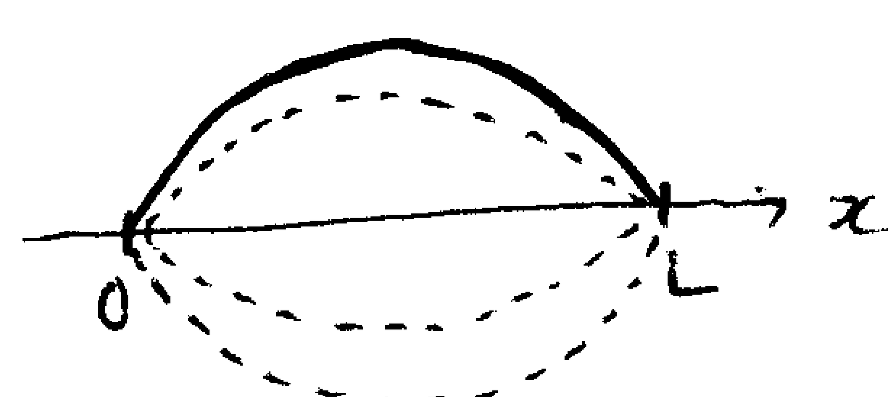
We can use the amplitude-phase form to write our solution to the wave equation as:

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\frac{an\pi t}{L} + B_n \sin\frac{an\pi t}{L} \right] \sin\frac{n\pi x}{L}$$

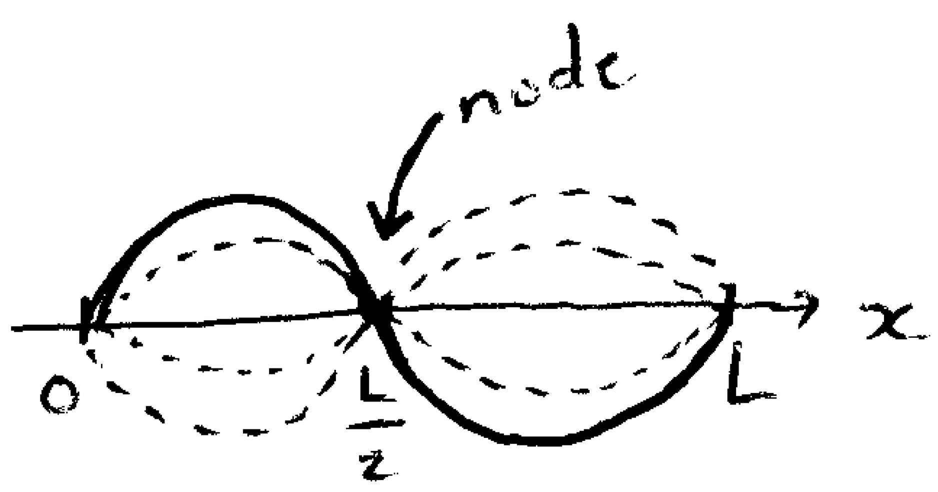
$$= \sum_{n=1}^{\infty} C_n \sin\left(\frac{an\pi t}{L} + \phi_n \right) \sin\frac{n\pi x}{L}$$

with $C_n = \sqrt{A_n^2 + B_n^2}$ and ϕ_n defined by $\sin\phi_n = \frac{A_n}{C_n}$, $\cos\phi_n = \frac{B_n}{C_n}$.

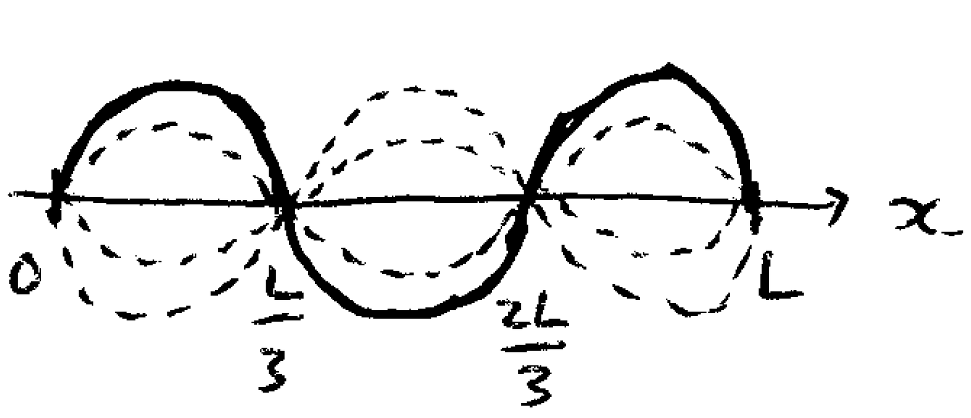
Now we see that $u(x,y)$ is a superposition of standing waves, a.k.a. modes, that is to say the graphs of $\sin(n\pi x/L)$ with time-varying amplitude $\left| C_n \sin\left(\frac{an\pi t}{L} + \phi_n\right) \right|$.

$n=1$ 

First standing wave (fundamental mode)

$n=2$ 

Second standing wave / mode ... 1 node

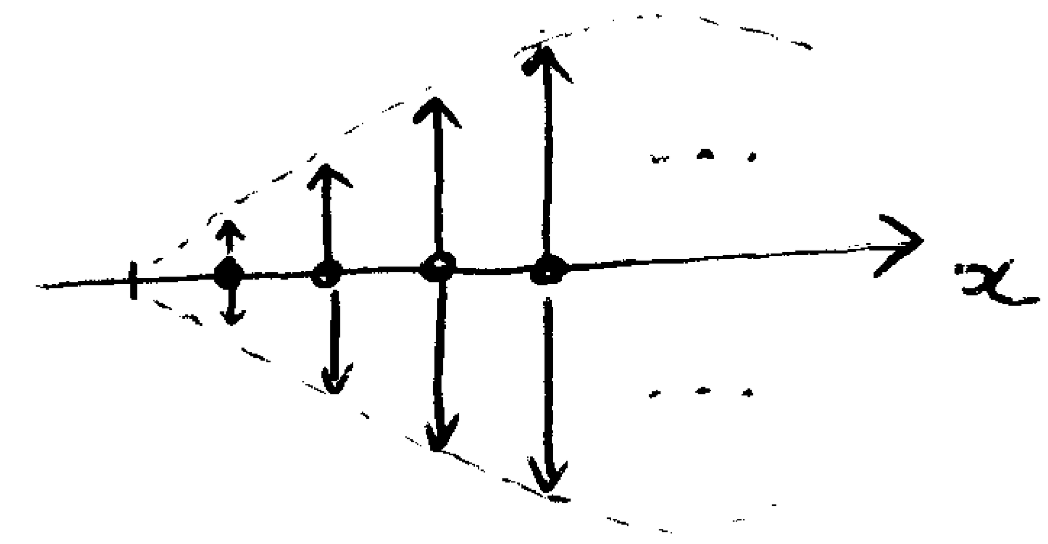
$n=3$ 

Third standing wave / mode --- 2 nodes

etc.

At a fixed value of x, a particular standing wave represents simple harmonic motion with amplitude $\left|c_n \sin\frac{n\pi x}{L}\right|$ and frequency $\frac{an}{2L}$.

$\therefore$ Each point on a standing wave vibrates with a different amplitude but with the same frequency.



For the first mode $(n=1)$, this frequency is

$$f_1 = \frac{a}{2L} = \boxed{\frac{1}{2L}\sqrt{\frac{T}{\rho}}}$$

recall : T is the tension and
ρ is the mass per unit length,
L is the length of the string.

We call f_1 the fundamental frequency (or first harmonic).
It is directly related to the pitch produced by a string in a musical instrument. Note, for example, that the greater the tension (T), the higher the pitch.

The frequencies of the other modes $(n = 2, 3, \ldots)$ are called overtones (2nd harmonic, 3rd harmonic, ...).

Example

Consider the specifications for an actual set of nylon guitar strings, shown on the next page ...

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	Diameter		Tension*	
Note	Inch	mm	Lbs	kg
E	.028	0.71	15.6	7.08
B	.032	0.81	11.3	5.12
G	.040	1.02	11.5	5.23
D	.030	0.76	16.7	7.55
A	.035	0.77	15.0	6.79
E	.043	1.09	13.8	6.27

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*25" (635mm) scale length used for tension measurements

2 Player's Points™

The density of nylon is $1010\ kg/m^3$.

Let's compute the fundamental frequency of the G-string.

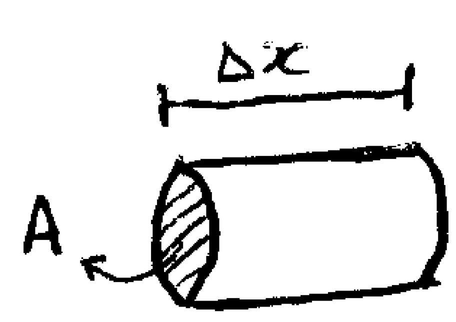
(Its actual frequency, according to standard tuning, is 196.00 Hz.)

$$f_1 = \frac{1}{2L}\sqrt{\frac{T}{\rho}}$$

Length of string : $L = 0.635\ m$

Tension in string : $T = 5.23\,kg \times 9.81\,ms^{-2} = 51.3\ N$

Mass per unit length :



$$\rho = \frac{A\,\Delta m}{A\,\Delta x} = \left(\frac{\Delta m}{\Delta V}\right)A = 1010 \times \pi\left(\frac{1.02\times10^{-3}}{2}\right)^2\ kg\,m^{-1}$$

density

$$\therefore f_1 = 196.32\ s^{-1} \approx 196\ Hz.$$