TW364 Lecture 12

12.4 WAVE EQUATION

Consider a vibrating string stretched between two points. Let $u(x,t)$ be the perpendicular displacement of the string at position x, time t.

The boundary value problem to be solved:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \qquad \text{with } a^2 = \frac{T}{\rho}$$

T ← tension in string

ρ ← string's mass per unit length

boundary conditions: $u(0,t) = 0$, $u(L,t) = 0$ [endpoints of string fixed]

initial conditions: $u(x,0) = f(x)$, $u_t(x,0) = g(x)$ [initial displacement, initial velocity]

<u>Solution</u>

Assume the product solution: $u(x,t) = X(x)T(t)$

Sub that into PDE: $XT'' = a^2 X''T \Rightarrow \frac{X''}{X} = \frac{T''}{a^2 T} = -\lambda$

$$\therefore \; X'' + \lambda X = 0$$

$$T'' + a^2 \lambda T = 0$$

Boundary conditions : $u(0,t)=0 \Rightarrow X(0)T(t)=0 \Rightarrow X(0)=0$

$u(L,t)=0 \Rightarrow X(L)T(t)=0 \Rightarrow X(L)=0$

Thus X can be determined by solving the Sturm-Liouville eigenvalue problem :

$$X'' + \lambda X = 0 \, , \quad X(0)=0 \, ; \quad X(L)=0$$

Once again, we make use of the result in Lecture 7 :

Eigenvalues : $\lambda_n = \dfrac{n^2\pi^2}{L^2}$, $n = 1, 2, \ldots$

Eigenfunctions : $X_n = c_1 \sin \dfrac{n\pi x}{L}$, $n = 1, 2, \ldots$

We solve T_n from $T_n'' + a^2\lambda_n T_n = 0$ (note : $a^2\lambda_n > 0$) :

$$T_n = c_2 \cos(a\sqrt{\lambda_n}\, t) + c_3 \sin(a\sqrt{\lambda_n}\, t)$$

$$= c_2 \cos \frac{an\pi t}{L} + c_3 \sin \frac{an\pi t}{L}$$

The superposition principle allows us to form

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos \frac{an\pi t}{L} + B_n \sin \frac{an\pi t}{L} \right] \sin \frac{n\pi x}{L} .$$

Set $t=0$, and enforce the 1st initial condition : $u(x,0) = f(x)$

$$f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}$$

That's the sine series of $f(x)$, so

$$A_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$$

Differentiate $u(x,t)$ with respect to t:

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\left[-A_n\left(\frac{an\pi}{L}\right)\sin \frac{an\pi t}{L} + B_n\left(\frac{an\pi}{L}\right)\cos \frac{an\pi t}{L}\right]\sin \frac{n\pi x}{L}$$

Set $t=0$, and enforce the 2^{nd} initial condition: $u_t(x,0) = g(x)$

$$g(x) = \sum_{n=1}^{\infty} B_n\left(\frac{an\pi}{L}\right)\sin \frac{n\pi x}{L}$$

That's the sine series of $g(x)$, so

$$B_n\left(\frac{an\pi}{L}\right) = \frac{2}{L}\int_0^L g(x)\sin \frac{n\pi x}{L}\, dx$$

$$B_n = \frac{2}{an\pi}\int_0^L g(x)\sin \frac{n\pi x}{L}\, dx .$$

To summarize, the solution to our BVP is:

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n \cos \frac{an\pi t}{L} + B_n \sin \frac{an\pi t}{L}\right]\sin \frac{n\pi x}{L},$$

with $A_n = \frac{2}{L}\int_0^L f(x)\sin \frac{n\pi x}{L}\, dx$

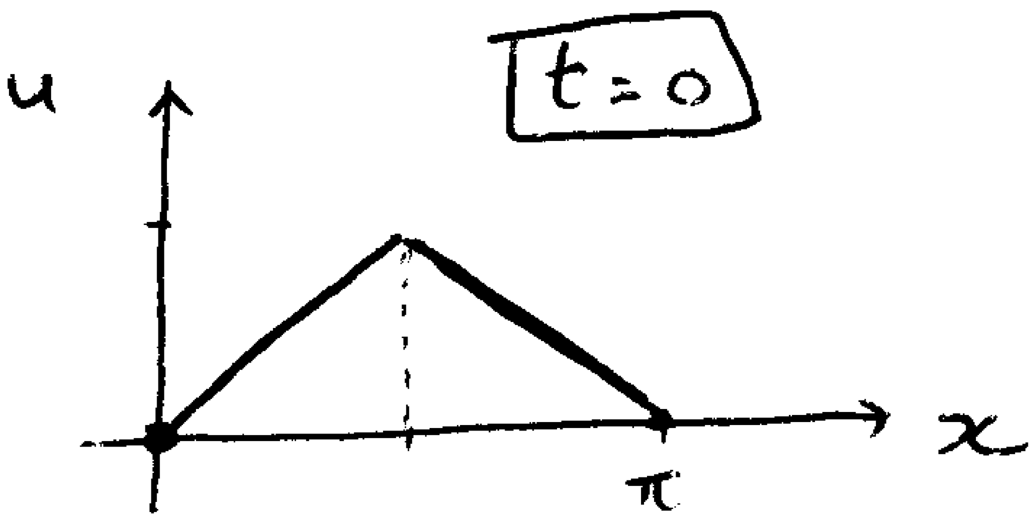
$$B_n = \frac{2}{an\pi}\int_0^L g(x)\sin \frac{n\pi x}{L}\, dx$$

Example

Suppose $L = \pi$

$a^2 = 1$

$$u(x,0) = \begin{cases} x, & 0 < x < \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \leq x < \pi \end{cases}$$



$u_t(x,0) = 0$ [we release the string from rest]

$$A_n = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} x \sin nx \, dx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x) \sin nx \, dx\right]$$

$$= \frac{4 \sin \frac{\pi n}{2}}{n^2 \pi}$$

$$B_n = \frac{2}{n\pi} \int_0^{\pi} 0 \cdot \sin nx \, dx$$

$$= 0$$

$$\therefore \quad u(x,t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \left(\frac{\sin \frac{\pi n}{2}}{n^2} \right) \cos nt \sin nx .$$