TW364 Lecture 11

12.3 HEAT EQUATION

Consider a thin rod of length L with initial temperature $f(x)$, $0 < x < L$, and whose ends are held at temp. <u>zero</u> for $t > 0$.

We ask for the temp. through the rod at time t, so we must solve:
$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

subject to: $u(0,t) = 0$; $u(L,t) = 0$; $u(x,0) = f(x)$.

<u>Solution</u>

Assume $u(x,t) = X(x)T(t)$: $XT' = kX''T$

Separate variables: $\dfrac{X''}{X} = \dfrac{T'}{kT} = -\lambda$

$$\therefore \; X'' + \lambda X = 0$$
$$T' + k\lambda T = 0$$

Boundary conditions:
$$u(0,t) = 0 \Rightarrow X(0)\,T(t) = 0$$
$$u(L,t) = 0 \Rightarrow X(L)\,T(t) = 0$$

Must hold for any t, so it is fair to assume
$$X(0) = 0, \quad X(L) = 0.$$

Thus X is a solution to the Sturm-Liouville problem:

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X(L) = 0.$$

In Lecture 7 we found the eigenvalues of this particular problem to be $\lambda_n = n^2\pi^2/L^2$, with corresponding non-trivial solutions:

$$X_n(x) = A_n \sin\frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots$$

We solve for T from $T' = -k\lambda T$:

$$T_n(t) = B_n e^{-k\left(\frac{n^2\pi^2}{L^2}\right)t}, \quad n = 1, 2, 3, \ldots$$

Hence, for some constant C_n, we have

$$u_n(x,y) = X_n(x)\,T_n(t) = C_n e^{-k\left(\frac{n^2\pi^2}{L^2}\right)t} \sin\frac{n\pi x}{L}.$$

Since the PDE is linear and homogeneous, we may use the superposition principle to form

$$u(x,y) = \sum_{n=1}^{\infty} C_n e^{-k\left(\frac{n^2\pi^2}{L^2}\right)t} \sin\frac{n\pi x}{L}$$

We have yet to enforce the initial condition $u(x,0) = f(x)$. The idea is to choose the constants $c_1, c_2, \ldots$ for this purpose.

$$u(x,0) = f(x) \quad : \quad f(x) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}.$$

This expression is exactly the sine series of $f(x)$!

The constants c_n are the coefficients of this series:

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \qquad n = 1, 2, \ldots$$

Finally, the solution to our BVP:

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx\right]e^{-k\left(\frac{n^2\pi^2}{L^2}\right)t}\sin\frac{n\pi x}{L}.$$

Note that $u(x,t) \to 0$ as $t \to \infty$ for all $x \in [0, L]$.

In this specific case (where the ends are held at <u>zero</u>) the steady state temperature throughout the rod is zero, as expected.

<u>Example</u>

Suppose $L = \pi$, $k = 1$ and the initial temp. is $f(x) = 100$.

Then $c_n = \frac{2}{\pi}\int_0^{\pi} 100\sin nx\,dx = \frac{200}{\pi}\left[\frac{1-(-1)^n}{n}\right]$

$$\therefore \quad u(x,t) = \frac{200}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n}\right]e^{-n^2 t}\sin nx.$$

We can plot the solution as a 3D surface over the xt-plane, or as curves over x for various fixed values of t.

Note: for these plots we compute partial sum approximations, with sufficiently large N.