TW364 Lecture 10

12.2 CLASSICAL PDEs

We will study the following three classical equations from physics (note: all are linear 2nd-order homogeneous PDEs):

1. HEAT equation : $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$, $k>0$ $\quad [u = u(x,t)]$

2. WAVE equation : $\dfrac{\partial^2 u}{\partial t^2} = a^2\dfrac{\partial^2 u}{\partial x^2}$ $\quad [u = u(x,t)]$

3. LAPLACE's equation : $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ $\quad [u = u(x,y)]$

It is straightforward to verify that:

the heat equation is parabolic,
the wave equation is hyperbolic,
Laplace's equation is elliptic.

We sometimes make use of the following notation, for neatness:

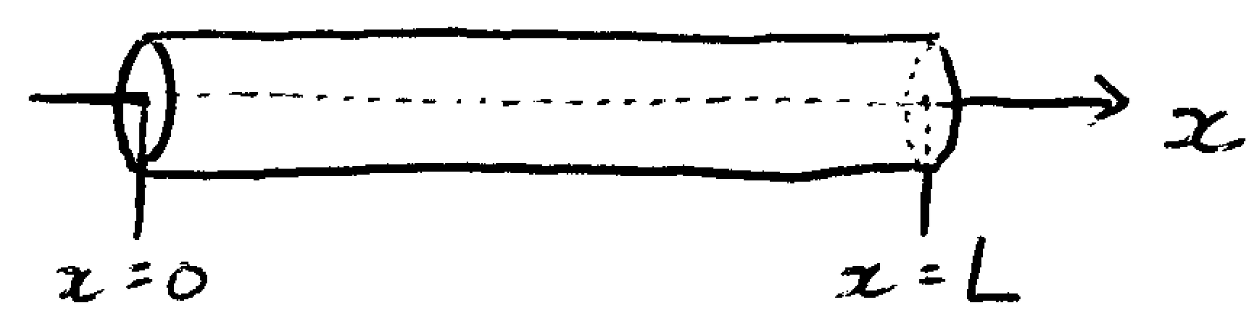
$$u_x = \frac{\partial u}{\partial x}, \quad u_y = \frac{\partial u}{\partial y}, \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad u_{xy} = \frac{\partial^2 u}{\partial x \partial y}, \quad \text{etc.}$$

Thus the 3 equations above can be written as:

$$u_t = k u_{xx} \quad ; \quad u_{tt} = a^2 u_{xx} \quad ; \quad u_{xx} + u_{yy} = 0 .$$

Heat equation

Special case of the "diffusion equation", used to model 1-D heat flow in a rod or thin wire.



The function $u(x,t)$ represents the temperature at point x in the rod at time t.

Under a few simplifying assumptions we get (see p. 461)

$$\boxed{\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}} \qquad k > 0$$

"thermal diffusivity" constant

Boundary conditions :

$$u(0,t) = u_0, \quad t > 0$$
$$u(L,t) = u_L, \quad t > 0$$

[We fix the temperature of the rod at its ends.]

Initial condition :

$$u(x,0) = f(x), \quad 0 < x < L$$

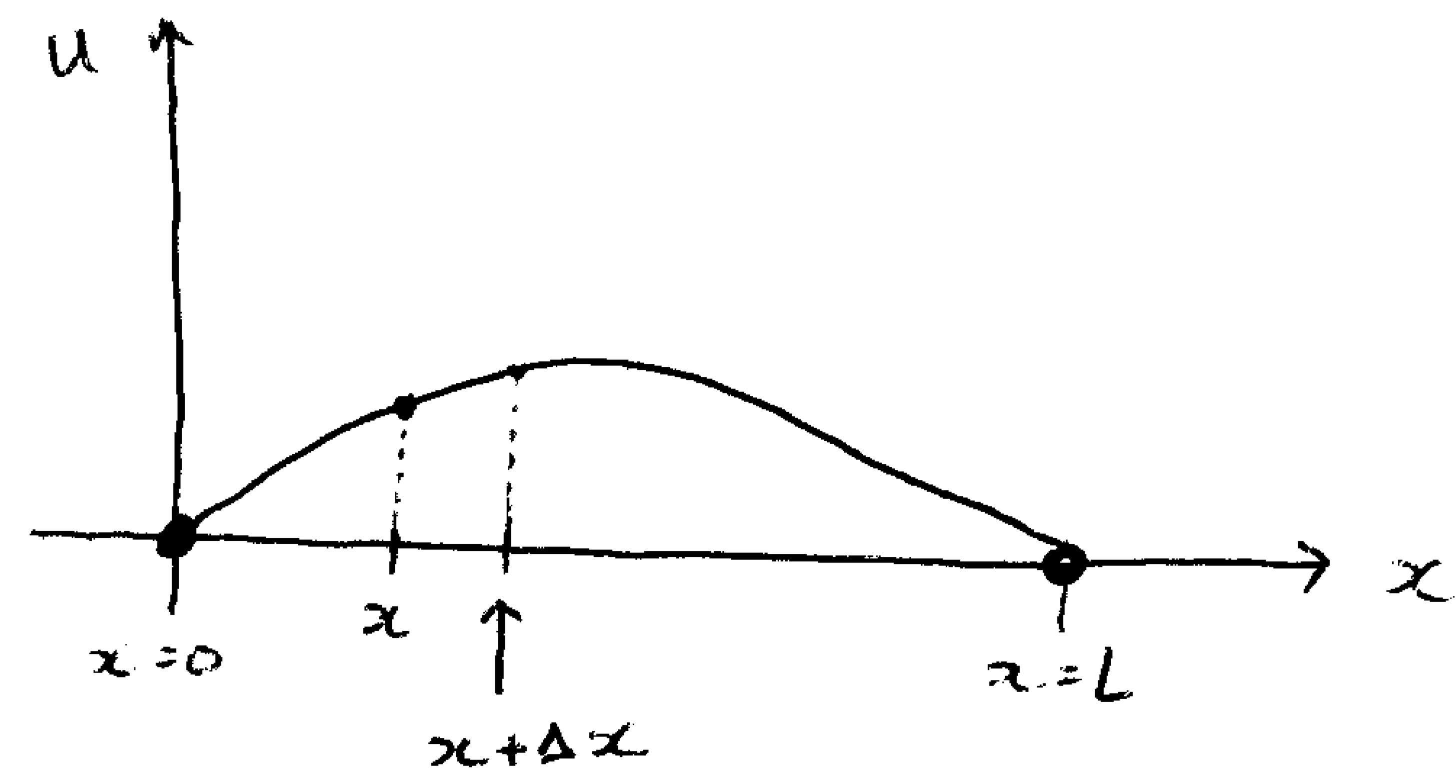
[We also set the initial temp. through the rod.]

As time goes by, heat flows through the rod, and the temperature changes and converges to a steady state $(t \to \infty)$.

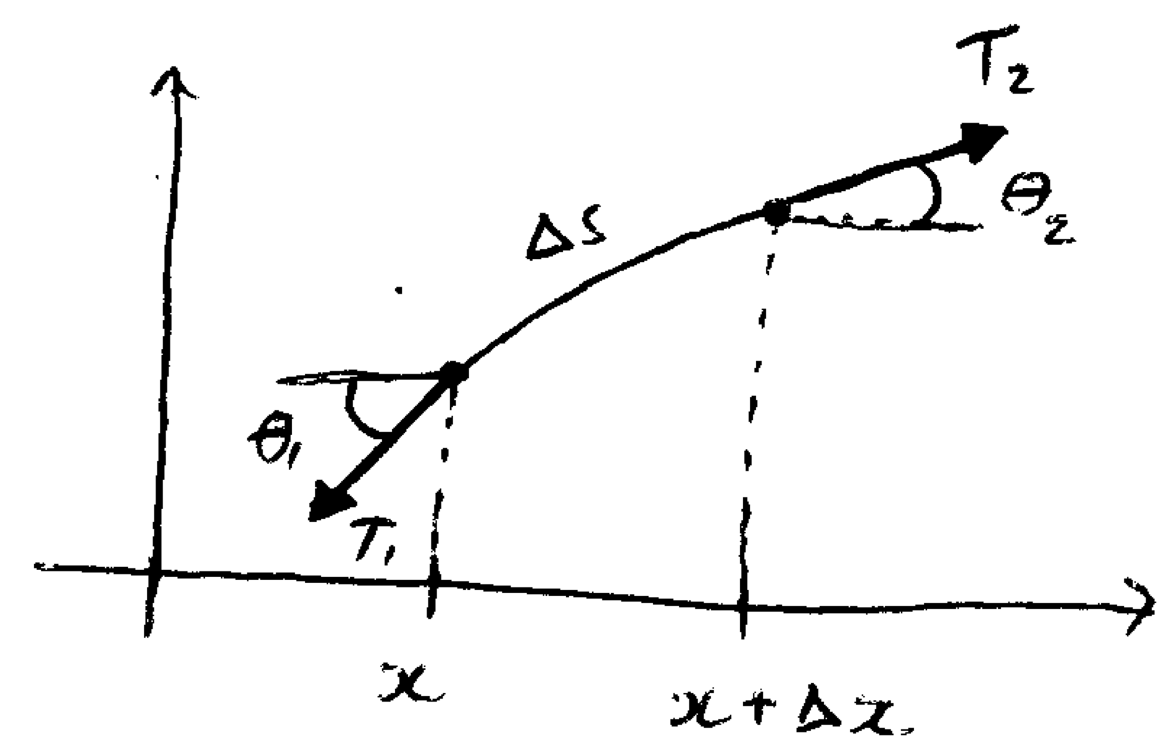
Wave equation

Models the vibration over time of a string (e.g. guitar string) stretched taut between two points.

Let $u(x,t)$ represent the vertical displacement of the string at position x, time t.



Consider a segment of the string, with mass Δm and length Δs.



Forces on this segment:

tension to the left (T_1) and tension to the right (T_2).

We assume T_1 and T_2 are large enough so that gravity is negligible.

Newton's 2nd law (in the vertical direction):

$$\Delta m \frac{\partial^2 u}{\partial t^2} = T_2 \sin\theta_2 - T_1 \sin\theta_1$$

Assume uniform tension ($T_1 = T_2 = T$) and small displacement ($\sin\theta_1 \approx \tan\theta_1$, $\sin\theta_2 \approx \tan\theta_2$):

$$\Delta m \frac{\partial^2 u}{\partial t^2} = T(\tan\theta_2 - \tan\theta_1) = T\left[u_x(x+\Delta x, t) - u_x(x,t)\right]$$

Let $\rho = \frac{\Delta m}{\Delta s}$ be the (uniform) mass per unit length,

and, with $\Delta x \ll 1$, we have $\rho \approx \frac{\Delta m}{\Delta x}$.

$$\rho \frac{\partial^2 u}{\partial t^2} = T \frac{u_x(x+\Delta x, t) - u_x(x,t)}{\Delta x}$$

Let $\Delta x \to 0$: $$\boxed{\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}}$$ with $a^2 = \frac{T}{\rho}$.

Boundary conditions :

$u(0,t) = 0 \quad , \quad t > 0$

$u(L,t) = 0 \quad , \quad t > 0$

[We fix the endpoints of the string.]

Initial conditions :

$u(x,0) = f(x)$

$u_t(x,0) = g(x)$

[We give the string an initial displacement and velocity.]

Laplace's equation

Occurs in time-independent problems involving potentials (e.g. electrostatic, gravitational and velocity in fluids). Also used to model steady state temperature distribution $u(x,y)$ over a plate (two spatial variables x and y).

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

... boundary conditions ...