TW364 Lecture 9

12.1 SEPARABLE PDE^s

General form of a linear 2^{nd}-order partial differential equation (PDE):

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

$A, B, \ldots, G$ are functions of x and y, and the PDE is <u>homogeneous</u> if $G(x,y) = 0$.

Solution: a function $u(x,y)$ possessing all the necessary partial derivatives that satisfies the equation in some region of the xy-plane.

We will be interested in <u>particular</u> solutions (finding general solutions of linear PDE^s is often difficult and, as it turns out, not all that useful in applications).

Separation of variables

Assume a solution in <u>product form</u>: $u(x,y) = X(x)\,Y(y)$.

This assumption can lead to a solution (not always!).

With the assumption $u(x,y) = XY$, we note that

$$\frac{\partial u}{\partial x} = X'Y \; ; \quad \frac{\partial u}{\partial y} = XY' \; ; \quad \frac{\partial^2 u}{\partial x^2} = X''Y \; ; \quad \frac{\partial^2 u}{\partial y^2} = XY''$$

Example

Find product solutions of $\dfrac{\partial^2 u}{\partial x^2} = 4\dfrac{\partial u}{\partial y}$

Assume $u = XY$: $X''Y = 4XY'$

separate variables : $\dfrac{X''}{4X} = \dfrac{Y'}{Y}$

Since the LHS is independent of y, and the RHS indep. of x, we conclude that both sides must be indep. of x <u>and</u> y.

$\therefore \dfrac{X''}{4X} = \dfrac{Y'}{Y} = -\lambda$, where λ is a constant

(the minus is convenient; not necessary).

Thus we have two ordinary DE^s :

$$X'' + 4\lambda X = 0 \quad \text{and} \quad Y' + \lambda Y = 0$$

* Suppose $\lambda = 0$:

$X'' = 0 \Rightarrow X = c_1 x + c_2$

$Y' = 0 \Rightarrow Y = c_3$

$\therefore u(x,y) = XY = (c_1 x + c_2)c_3 = A_1 x + B_1$ ①

* Suppose $\lambda < 0$, say $\lambda = -\alpha^2$:

$$X'' = 4\alpha^2 X \quad \Rightarrow \quad X = c_4 \cosh 2\alpha x + c_5 \sinh 2\alpha x$$

$$Y' = \alpha^2 Y \quad \Rightarrow \quad Y = c_6 e^{\alpha^2 y}$$

$$\therefore \; u(x,y) = (c_4 \cosh 2\alpha x + c_5 \sinh 2\alpha x)\, c_6 e^{\alpha^2 y}$$

$$= A_2 e^{\alpha^2 y} \cosh 2\alpha x + B_2 e^{\alpha^2 y} \sinh 2\alpha x \qquad (2)$$

* Suppose $\lambda > 0$, say $\lambda = \alpha^2$:

$$X'' = -4\alpha^2 X \quad \Rightarrow \quad X = c_7 \cos 2\alpha x + c_8 \sin 2\alpha x$$

$$Y' = -\alpha^2 Y \quad \Rightarrow \quad Y = c_9 e^{-\alpha^2 y}$$

$$\therefore \; u(x,y) = (c_7 \cos 2\alpha x + c_8 \sin 2\alpha x)\, c_9 e^{-\alpha^2 y}$$

$$= A_3 e^{-\alpha^2 y} \cos 2\alpha x + B_3 e^{-\alpha^2 y} \sin 2\alpha x \qquad (3)$$

It is easily verified that (1), (2) and (3) are all solutions of the given PDE.

Superposition principle

If $u_1, u_2, \ldots, u_k$ are solutions of a homogeneous linear PDE, then any linear combination,

$$u = c_1 u_1 + c_2 u_2 + \cdots + c_k u_k,$$

where $c_1, c_2, \ldots, c_k$ are constants, is also a solution.

Classification of PDE^s

The linear 2^{nd}-order PDE

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G,$$

where $A, B, \ldots, G$ are real constants, is said to be

* hyperbolic if $B^2 - 4AC > 0$,
* parabolic if $B^2 - 4AC = 0$,
* elliptic if $B^2 - 4AC < 0$.

Examples

(a) $3\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial y}$: $A = 3,\ B = 0,\ C = 0 \Rightarrow B^2 - 4AC = 0$

This PDE is parabolic.

(b) $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial y^2}$: $A = 1,\ B = 0,\ C = -1 \Rightarrow B^2 - 4AC > 0$

This PDE is hyperbolic.

(c) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$: $A = 1,\ B = 0,\ C = 1 \Rightarrow B^2 - 4AC < 0$

This PDE is elliptic.