TW364 Lecture 8

In the previous lecture (and Tutorial 2) we looked at a special case of the following two-point boundary value problem, called the Sturm-Liouville problem:

$$\text{Solve} \quad \frac{d}{dx}\left[r(x)\, y'\right] + \left[q(x) + \lambda p(x)\right] y = 0 \qquad \begin{pmatrix} r(x) > 0 \\ p(x) > 0 \end{pmatrix}$$

$$\text{subject to} \quad A_1 y(a) + B_1 y'(a) = 0 \qquad [\text{the condition at } x = a]$$

$$A_2 y(b) + B_2 y'(b) = 0 \qquad [\text{the condition at } x = b]$$

Example

The BVP: $y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0$

is a Sturm-Liouville problem, with $r(x) = 1$, $q(x) = 0$, $p(x) = 1$, $a = 0$, $b = L$, $A_1 = 1$, $B_1 = 0$, $A_2 = 1$, $B_2 = 0$.

Clearly, Sturm-Liouville problems always possess the trivial solution: $y = 0$. But this is of no interest to us.

We seek numbers λ (eigenvalues) corresponding to non-trivial solutions y (eigenfunctions).

Properties of Sturm-Liouville problems

Theorem 11.4.1, p. 441 :

* There are infinitely many eigenvalues, and they can be arranged $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ such that $\lambda_n \to \infty$ as $n \to \infty$.

* For each eigenvalue there is one eigenfunction (up to scale), and the eigenfunctions are linearly independent.

* The set of eigenfunctions is orthogonal on $[a,b]$ with respect to the weight function $p(x)$.

Example

Find the non-trivial solutions of the Sturm-Liouville problem :

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) + y'(1) = 0.$$

* Suppose $\lambda = 0$:

$$y'' = 0 \Rightarrow \quad y = c_1 x + c_2$$
$$y' = c_1$$

$y(0) = 0$: $\quad c_1 \cdot 0 + c_2 = 0 \quad \Rightarrow c_2 = 0$

$y(1) + y'(1) = 0$: $\quad c_1 \cdot 1 + c_2 + c_1 = 0 \Rightarrow c_1 = 0$

$\therefore y = 0$ (trivial solution)

* Suppose $\lambda < 0$, say $\lambda = -\alpha^2$:

$$y'' = \alpha^2 y \quad \Rightarrow \quad y = c_1 \cosh \alpha x + c_2 \sinh \alpha x$$
$$y' = \alpha c_1 \sinh \alpha x + \alpha c_2 \cosh \alpha x$$

$y(0) = 0$: $c_1 \cdot 1 + c_2 \cdot 0 = 0 \Rightarrow c_1 = 0$

$y(1) + y'(1) = 0$: $c_1 \cosh \alpha + c_2 \sinh \alpha + \alpha c_1 \sinh \alpha + \alpha c_2 \cosh \alpha = 0$

$$c_2(\sinh \alpha + \alpha \cosh \alpha) = 0$$

We note that $f(\alpha) = \sinh \alpha + \alpha \cosh \alpha$ has only one zero, $\alpha = 0$, but α cannot be 0 (since $\lambda = -\alpha^2$ and $\lambda < 0$). So $c_2 = 0$.

$\therefore y = 0$ (trivial solution)

* Suppose $\lambda > 0$, say $\lambda = \alpha^2$:

$$y'' = -\alpha^2 y \quad \Rightarrow \quad y = c_1 \cos \alpha x + c_2 \sin \alpha x$$
$$y' = -\alpha c_1 \sin \alpha x + \alpha c_2 \cos \alpha x$$

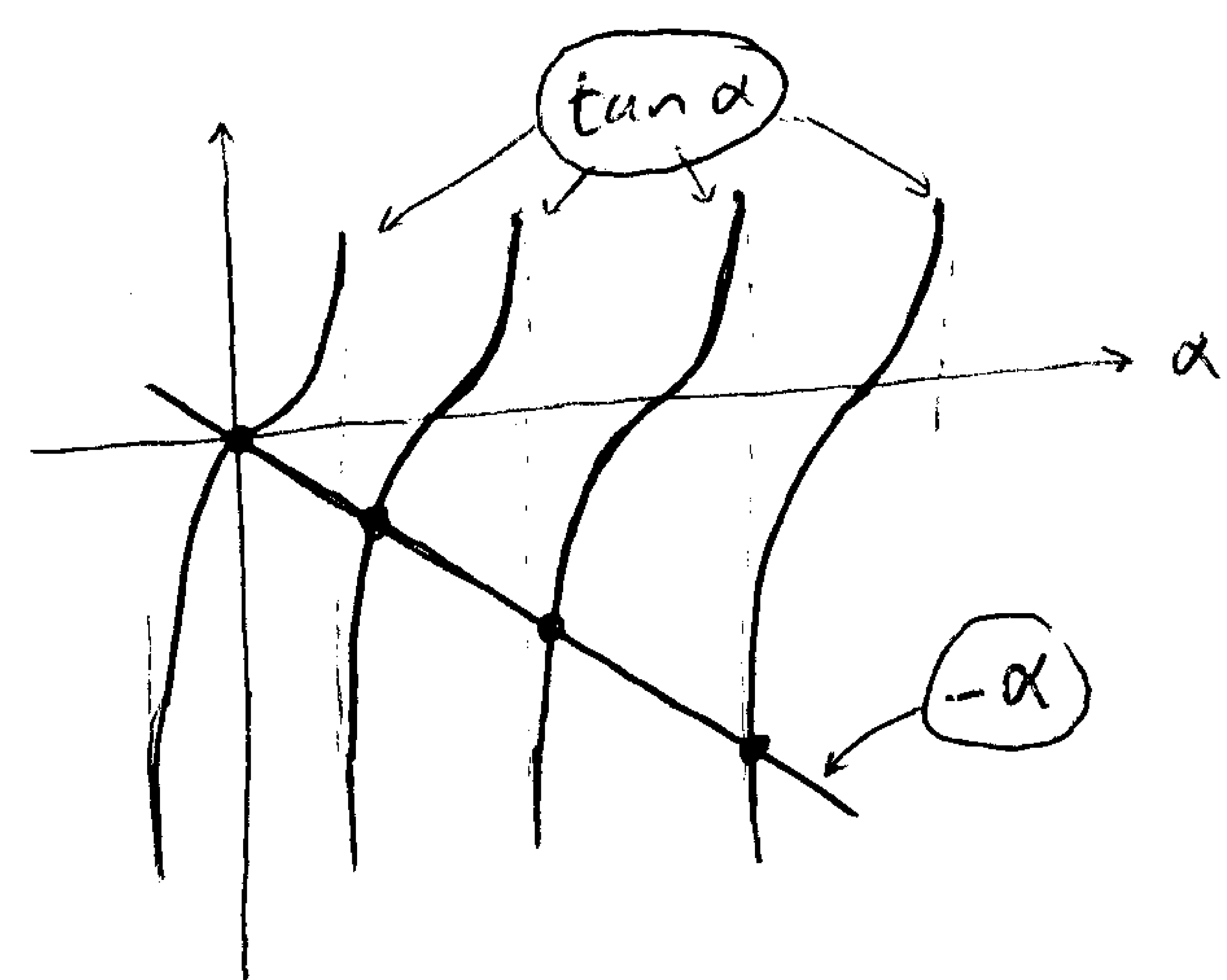
$y(0) = 0$: $c_1 \cdot 1 + c_2 \cdot 0 = 0 \Rightarrow c_1 = 0$

$y(1) + y'(1) = 0$: $c_1 \cos \alpha + c_2 \sin \alpha - \alpha c_1 \sin \alpha + \alpha c_2 \cos \alpha = 0$

$$c_2(\sin \alpha + \alpha \cos \alpha) = 0$$

For non-trivial solutions we demand $c_2 \neq 0$, so we must choose λ in such a way that

$$\sin \alpha + \alpha \cos \alpha = 0$$
$$\therefore \tan \alpha = -\alpha .$$



This equation has infinitely many solutions.

The eigenvalues are $\lambda_n = \alpha_n^2$, $n = 1, 2, 3, \ldots$, where $\alpha_1, \alpha_2, \alpha_3, \ldots$ are the positive roots of $\tan\alpha = -\alpha$

$$\alpha_1 = 2.0288\cdots$$
$$\alpha_2 = 4.9132\cdots$$
$$\alpha_3 = 7.9787\cdots$$
$$\alpha_4 = 11.0855\cdots \quad \text{etc.}$$

The eigenfunctions are $y_n = \sin\alpha_n x$, $n = 1, 2, 3, \ldots$