TW364 Lecture 7

11.4 STURM-LIOUVILLE PROBLEM

(First, some Applied Maths 244 revision...)

$\boxed{1.}$ Solve $\dfrac{d^2y}{dx^2} = \alpha^2 y$

Try $y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2 e^{mx}$

Substitute into DE: $m^2 e^{mx} = \alpha^2 e^{mx}$

$\therefore$ $m = \alpha$ or $m = -\alpha$

General solution: $y = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$

Alternative form, using hyperbolic trig functions:

$\cosh t = \frac{1}{2}(e^t + e^{-t})$, $\sinh t = \frac{1}{2}(e^t - e^{-t})$

$\therefore$ $e^t = \cosh t + \sinh t$, $e^{-t} = \cosh t - \sinh t$

$$y = c_1(\cosh \alpha x + \sinh \alpha x) + c_2(\cosh \alpha x - \sinh \alpha x)$$

$$= (c_1 + c_2)\cosh \alpha x + (c_1 - c_2)\sinh \alpha x$$

$$= C_1 \cosh \alpha x + C_2 \sinh \alpha x.$$

$\boxed{2.}$ Solve $\dfrac{d^2y}{dx^2} = -\alpha^2 y$

Try $y = e^{mx}$: $m^2 e^{mx} = -\alpha^2 e^{mx}$

$\therefore$ $m = i\alpha$ or $m = -i\alpha$

General solution :

$$y = c_1 e^{i\alpha x} + c_2 e^{-i\alpha x}$$

$$= c_1(\cos\alpha x + i\sin\alpha x) + c_2(\cos\alpha x - i\sin\alpha x)$$

$$= (c_1 + c_2)\cos\alpha x + i(c_1 - c_2)\sin\alpha x$$

$$= C_1 \cos\alpha x + C_2 \sin\alpha x.$$

Summary

$$\frac{d^2y}{dx^2} = \alpha^2 y \rightarrow y = c_1\cosh\alpha x + c_2\sinh\alpha x$$

$$\frac{d^2y}{dx^2} = -\alpha^2 y \Rightarrow y = C_1\cos\alpha x + C_2\sin\alpha x$$

Eigenvalues and eigenfunctions

Consider the following 2^{nd}-order boundary value problem :

$$y'' + \lambda y = 0, \quad y(0) = 0 \text{ and } y(L) = 0.$$

Of course, we have the trivial solution $y = 0$, but that's not interesting.

We ask: for which values of λ can we find non-trivial solutions?

* Suppose $\lambda = 0$:

$$y'' = 0 \Rightarrow y' = c_1 \Rightarrow y = c_1 x + c_2$$

Boundary conditions: $y(0) = 0$: $0 = c_1 \cdot 0 + c_2 \Rightarrow c_2 = 0$

$y(L) = 0$: $0 = c_1 L + 0 \rightarrow c_1 = 0$

$\therefore y = 0$ (trivial!)

* Suppose $\lambda < 0$, say $\lambda = -\alpha^2$:

$$y'' = \alpha^2 y \Rightarrow y = c_1 \cosh \alpha x + c_2 \sinh \alpha x$$

$y(0) = 0$: $0 = c_1 \underbrace{\cosh 0}_{=1} + c_2 \underbrace{\sinh 0}_{=0} \Rightarrow c_1 = 0$

$y(L) = 0$: $0 = 0 + c_2 \sinh \alpha L$

$\alpha \neq 0$ (since $\lambda = -\alpha^2 < 0$) $\Rightarrow c_2 = 0$

$\therefore y = 0$ (trivial!)

* Suppose $\lambda > 0$, say $\lambda = \alpha^2$:

$$y'' = -\alpha^2 y \Rightarrow y = c_1 \cos \alpha x + c_2 \sin \alpha x$$

...

$y(0) = 0: \quad 0 = c_1 \cdot 1 + c_2 \cdot 0 \quad \Rightarrow \quad c_1 = 0$

$y(L) = 0: \quad 0 = 0 + c_2 \sin \alpha L$

To avoid $c_2 = 0$ (which would lead to $y = 0$),

we must ensure that $\sin \alpha L = 0$

$\therefore \alpha L = n\pi$ where $n \in \mathbb{Z}$.

Since $\lambda = \alpha^2$, we may define

$$\lambda_n = \frac{n^2 \pi^2}{L^2}, \quad n = 1, 2, 3, \ldots$$

These values of λ that produce non-trivial solutions are called <u>eigenvalues</u>, and the corresponding solutions

$$y_n = c_2 \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots$$

are <u>eigenfunctions</u> of the particular BVP.

Note that, similar to eigenvectors in Linear Algebra, eigenfunctions are defined up to scale (the constant c_2 is arbitrary, as long as it's not zero).

The BVP $y'' + \lambda y = 0$, $y(0) = 0$; $y(L) = 0$ is a specific example of the more general <u>Sturm-Liouville</u> problem.