TW364 Lecture 6

In this lecture we look at some additional properties of Fourier series, not covered in the textbook.

Rate of convergence

Partial sum approximations : $f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{N-1} \left[a_n \cos \frac{n\pi x}{p} + b_n \sin \frac{n\pi x}{p} \right]$

How rapidly do these approximations tend to f ?
or, put differently, how rapidly do the Fourier coefficients approach 0 as $n \to \infty$?

If there are jump discontinuities in (the periodic extension of) f,

then $a_n, b_n \sim \frac{c}{n}$ (here we also get Gibbs oscillations)

If f is continuous but f' is discontinuous,

then $a_n, b_n \sim \frac{c}{n^2}$

If f and f' are continuous but f'' is discontinuous,

then $a_n, b_n \sim \frac{c}{n^3}$

etc.

Note: $f \sim g$ means $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 1$.

Parseval's theorem

If f and its periodic extension are continuous on $[-p, p]$, and f' is piecewise continuous, then

$$\frac{1}{p}\int_{-p}^{p}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

Proof :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi x}{p} + b_n\sin\frac{n\pi x}{p}\right]$$

$$[f(x)]^2 = \frac{a_0}{2}f(x) + \sum_{n=1}^{\infty}\left[a_n f(x)\cos\frac{n\pi x}{p} + b_n f(x)\sin\frac{n\pi x}{p}\right]$$

$$\int_{-p}^{p}[f(x)]^2dx = \frac{a_0}{2}\int_{-p}^{p}f(x)\,dx$$

$$+ \sum_{n=1}^{\infty}\left[a_n\int_{-p}^{p}f(x)\cos\frac{n\pi x}{p}\,dx + b_n\int_{-p}^{p}f(x)\sin\frac{n\pi x}{p}\,dx\right]$$

$$= \frac{a_0}{2}\cdot p\,a_0 + \sum_{n=1}^{\infty}\left[a_n\cdot p a_n + b_n\cdot p b_n\right]$$

$$\therefore \frac{1}{p}\int_{-p}^{p}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

Side note : in signal processing the quantity $\frac{1}{p}\int_{-p}^{p}[f(x)]^2dx$ is called the "energy" of the signal/wave f.

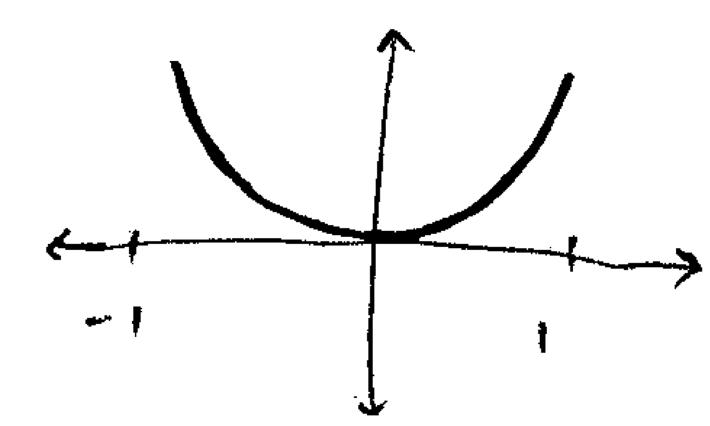
Among other applications, Parseval's theorem can be used in summing certain infinite series.

Example

Consider $f(x) = x^2, \quad 0 < x < 1$

The cosine series for the even expansion of f:

$$f(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2 \pi^2} \cos n\pi x.$$



Let's apply Parseval's theorem:

$$\int_{-p}^{p} [f(x)]^2 \, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

$$\int_{-1}^{1} (x^2)^2 \, dx = \frac{1}{2}\left(\frac{2}{3}\right)^2 + \sum_{n=1}^{\infty} \left[\left(\frac{4(-1)^n}{n^2\pi^2} \right)^2 + 0^2 \right]$$

$$\frac{1}{5} x^5 \Big|_{-1}^{1} = \frac{1}{2}\left(\frac{4}{9}\right) + \sum_{n=1}^{\infty} \frac{16}{n^4 \pi^4}$$

$$\frac{2}{5} = \frac{2}{9} + \frac{16}{\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Differentiation of Fourier Series

Example : consider the Fourier series of $f(x) = x$, $-\pi < x < \pi$:

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$$

$$x = 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots \right)$$

Differentiate both sides :

$$1 = 2\left(\cos x - \cos 2x + \cos 3x - \cdots \right)$$

Let $x = 0$:

$$\frac{1}{2} = 1 - 1 + 1 - 1 + \cdots \qquad ????$$

It can be shown that differentiation of the Fourier series can be performed safely (in the sense that the derivative of the series converges to f') only if f and its periodic extension are continuous on $[-p, p]$ and f' is piecewise continuous.

In the example above, the periodic extension of f is not continuous.

Integration of Fourier series is fine, as long as f is piecewise continuous (which it must be, for the Fourier series itself to converge to f).