TW364 Lecture 5

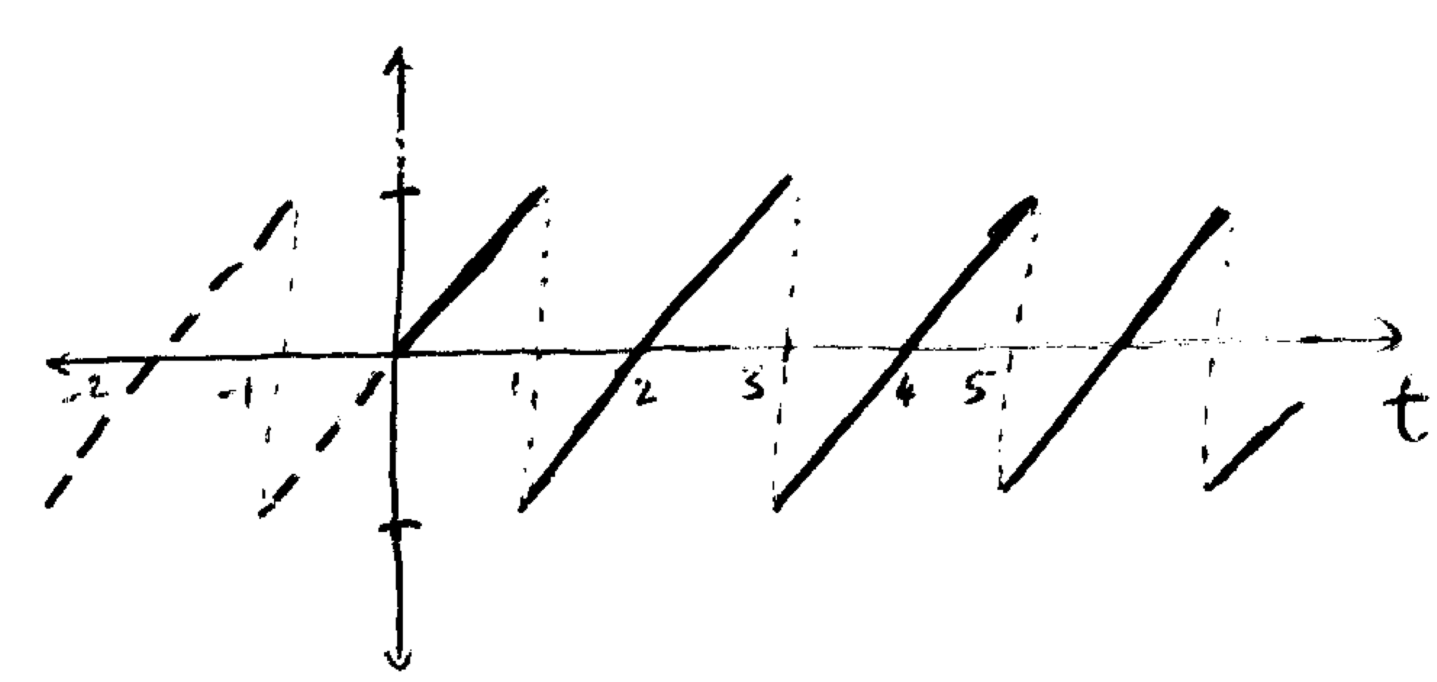
Periodic driving force

Fourier series can be useful in finding a solution of a differential equation (DE) describing a system with periodic driving force.

Recall the DE $\quad m\dfrac{d^2x}{dt^2} + kx = f(t), \quad (*)$

that describes an undamped spring-mass system with external (driving) force $f(t)$, object mass m, and spring constant k.

Suppose for example $m = \frac{1}{16}$, $k = 4$, and $f(t)$ is the periodic function shown on the right.



Here $f(t)$ is defined for $t \geq 0$, but we may extend it to the negative t-axis in a periodic manner (as shown).
Thus $f(t)$ is the odd expansion of πt, $0 < t < 1$, and we can express f as a sine series:

$$f(t) = \sum_{n=1}^{\infty} b_n \sin n\pi t$$

$$\text{where } b_n = 2\int_0^1 \pi t \sin n\pi t \, dt = \frac{2(-1)^{n+1}}{n}$$

So, for our example the DE in ⊛ becomes

$$\frac{1}{16}\frac{d^2x}{dt^2} + 4x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin n\pi t \qquad (**)$$

This DE is linear and non-homogeneous. By the method of undetermined coefficients we assume a particular solution similar in form to the R.H.S., i.e.

$$x_p = \sum_{n=1}^{\infty} \left[A_n \cos n\pi t + B_n \sin n\pi t \right]$$

$$x_p' = \sum_{n=1}^{\infty} \left[-A_n n\pi \sin n\pi t + B_n n\pi \cos n\pi t \right]$$

$$x_p'' = \sum_{n=1}^{\infty} \left[-A_n n^2\pi^2 \cos n\pi t - B_n n^2\pi^2 \sin n\pi t \right]$$

The L.H.S. of (**) is then

$$\frac{1}{16}x_p'' + 4x_p = \sum_{n=1}^{\infty} \left[\left(-\frac{n^2\pi^2}{16} + 4\right) A_n \cos n\pi t + \left(-\frac{n^2\pi^2}{16} + 4\right) B_n \sin n\pi t \right]$$

and we want it to equal the R.H.S. of (**), so

$$\left(-\frac{n^2\pi^2}{16} + 4\right) A_n = 0 \quad \Rightarrow \quad A_n = 0$$

$$\left(-\frac{n^2\pi^2}{16} + 4\right) B_n = \frac{2(-1)^{n+1}}{n} \quad \Rightarrow \quad B_n = \frac{32(-1)^{n+1}}{n(64 - n^2\pi^2)}$$

Hence, a particular solution for (**) is

$$x_p = \sum_{n=1}^{\infty} \frac{32(-1)^{n+1}}{n(64 - n^2\pi^2)} \sin n\pi t .$$

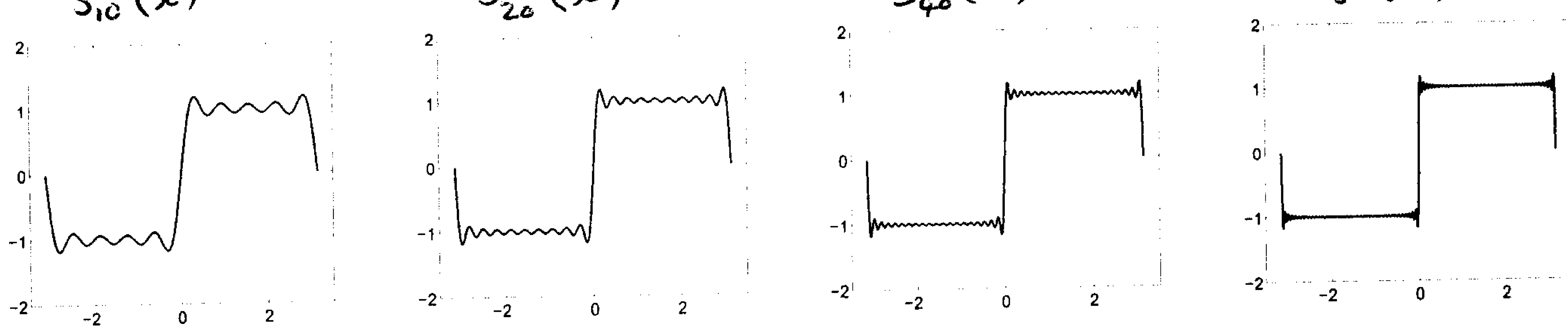
The Gibbs phenomenon

In Lecture 4 we found the Fourier series of $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 \le x < \pi \end{cases}$

to be $f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin nx$.

Here are a few partial sum approximations :

$S_{10}(x)$ $S_{20}(x)$ $S_{40}(x)$ $S_{80}(x)$



The oscillations exhibited by these approximations around function discontinuities (also in the periodic extension of f) are referred to as the Gibbs phenomenon.

The overshoot in S_N does not die out as $N \to \infty$, but approaches a finite limit. It turns out that this limit is about 8.95% of the magnitude of the particular jump discontinuity.

For the example above, the function's discontinuity is 2,

and $\frac{8.95}{100} \times 2 = 0.179$.

If the Gibbs overshoot does not vanish in the limit, does that not contradict the convergence theorem of Fourier series (Theorem 11.2.1, page 428) ?

No! The theorem describes point-wise convergence (for a fixed x, where $f(x)$ is continuous, the Fourier series converges to $f(x)$).

Since the region where oscillations occur gets narrower as N (number of terms in the partial sum) increases, the oscillations will disappear in the neighbourhood of any x where $f(x)$ is continuous, even arbitrarily close to the discontinuity.