TW364 Lecture 4

11.3 FOURIER COSINE AND SINE SERIES

Even and odd functions

- a function f is even if $f(-x) = f(x)$
- a function f is odd if $f(-x) = -f(x)$

Examples: $f(x) = x^2$ is even

$f(x) = x^3$ is odd

$f(x) = e^x$ is neither even nor odd

Note that

$\cos kx$ is even	for any $k \in \mathbb{R}$
$\sin kx$ is odd	for any $k \in \mathbb{R}$

Properties:

- the product of two even functions is even
- the product of two odd functions is even
- the product of an even and odd function is odd
- the sum (difference) of two even functions is even
- the sum (difference) of two odd functions is odd
- if f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$
- if f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$

If f is <u>even</u> on $(-p,p)$, its Fourier coefficients become

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx = \frac{2}{p}\int_0^p f(x)\,dx$$

$$a_n = \frac{1}{p}\int_{-p}^{p} \underbrace{f(x)\cos\frac{n\pi x}{p}}_{\text{even}}\,dx = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi x}{p}\,dx$$

$$b_n = \frac{1}{p}\int_{-p}^{p} \underbrace{f(x)\sin\frac{n\pi x}{p}}_{\text{odd}}\,dx = 0$$

$\therefore$ for an <u>even</u> function f we get the <u>cosine series</u>:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{p},$$

where $a_0 = \frac{2}{p}\int_0^p f(x)\,dx$, $\quad a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi x}{p}\,dx$

If f is <u>odd</u> on $(-p,p)$,

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx = 0$$

$$a_n = \frac{1}{p}\int_{-p}^{p} \underbrace{f(x)\cos\frac{n\pi x}{p}}_{\text{odd}}\,dx = 0$$

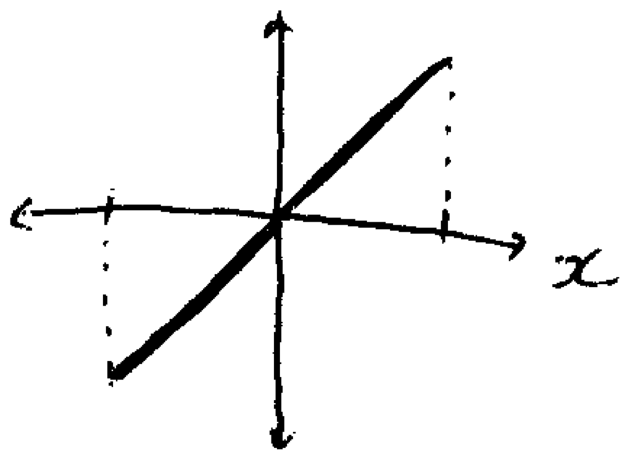
$$b_n = \frac{1}{p}\int_{-p}^{p} \underbrace{f(x)\sin\frac{n\pi x}{p}}_{\text{even}}\,dx = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi x}{p}\,dx$$

$\therefore$ for an <u>odd</u> function f we get the <u>sine series</u>:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{p}, \qquad \text{where } b_n = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi x}{p}\,dx$$

Example

Find the Fourier series of $f(x) = x, \quad -2 < x < 2$.



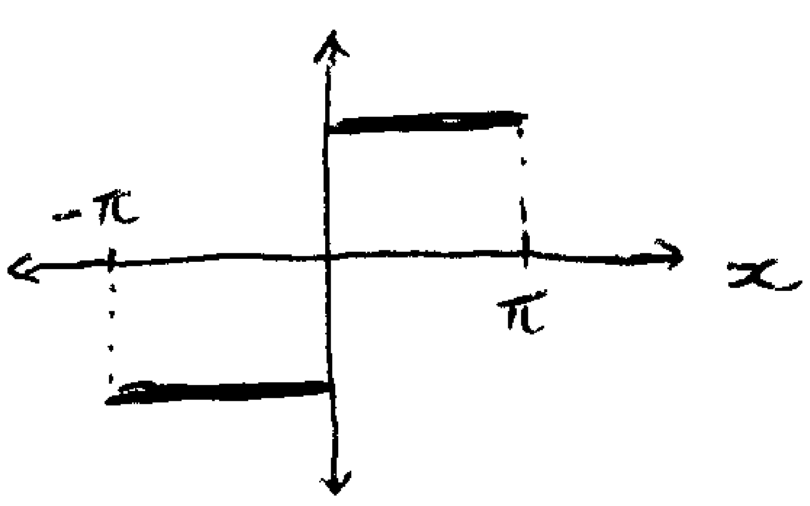
We note that this function is odd, so we expand in a <u>sine</u> series.

$$b_n = \int_0^2 x \sin \frac{n\pi x}{2}\, dx = \frac{4(-1)^{n+1}}{n\pi}$$

$$\therefore \ f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{2} .$$

Example

Find the Fourier series of $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 2, & 0 \leq x < \pi \end{cases}$



We note that $f(x) - 1$ is odd, so we expand $f(x) - 1$ in a <u>sine</u> series.

$$b_n = \frac{2}{\pi} \int_0^{\pi} \left(f(x) - 1 \right) \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi} \sin nx \, dx = \frac{2 - 2(-1)^n}{n\pi}$$

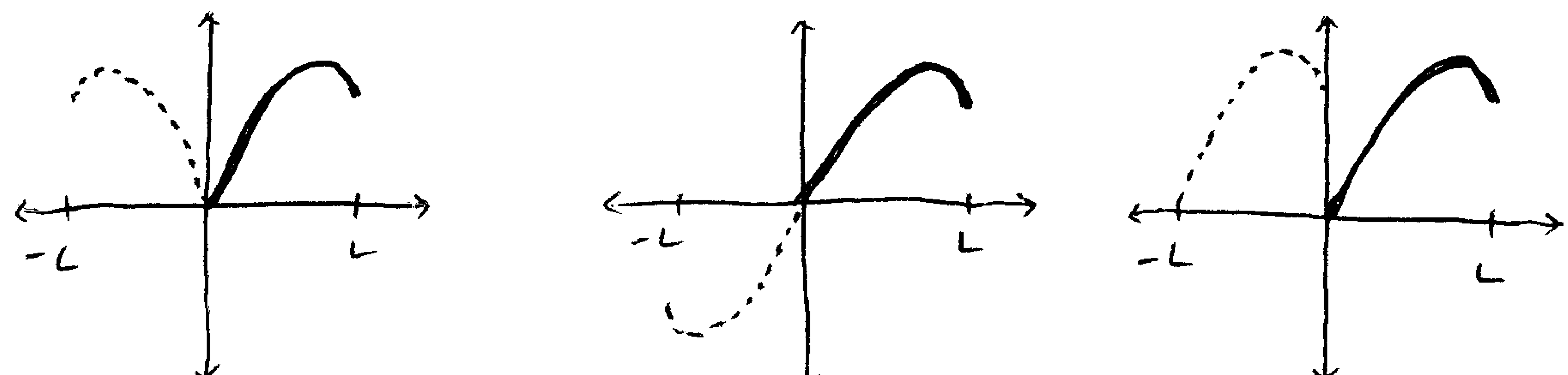
$$f(x) - 1 = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \sin nx$$

$$\therefore \ f(x) = 1 + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \sin nx$$

Half-range expansions

Suppose a function f is defined on $[0, L]$.
We may now choose a definition of f on $[-L, 0)$, in order to find its Fourier series.

Some options:



even reflection	odd reflection	identity reflection
$\Rightarrow$ cosine series	$\Rightarrow$ sine series	$\Rightarrow$ Fourier series

Each of these will produce a different series expansion, with different periodic extensions, but all will converge to f on the original interval of interest, namely $[0, L]$.

Example

Expand $f(x) = x^2, \quad 0 < x < \pi$ in

(a) a cosine series

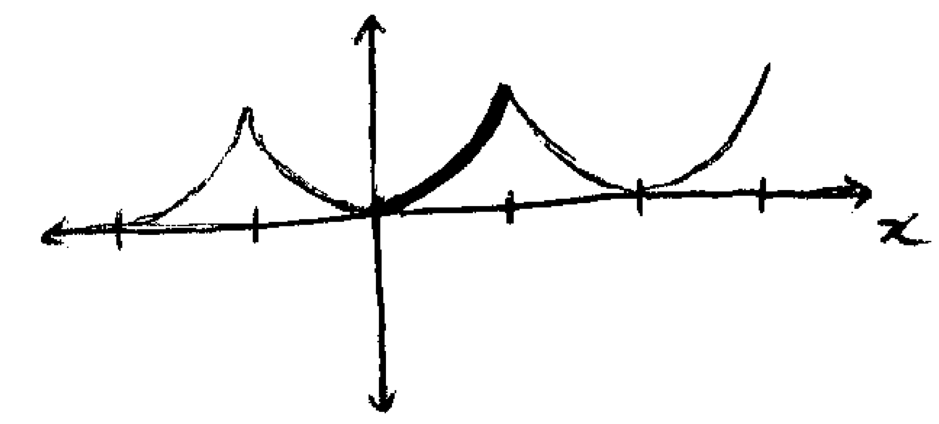
(b) a sine series

(c) a Fourier series

(a) $$a_0 = \frac{2}{\pi}\int_0^\pi x^2\,dx = \frac{2\pi^2}{3}$$

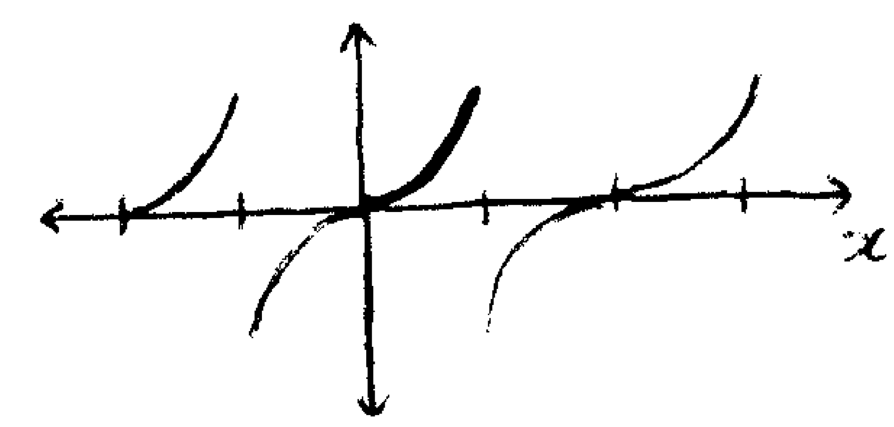
$$a_n = \frac{2}{\pi}\int_0^\pi x^2 \cos nx\,dx = \frac{4(-1)^n}{n^2}$$

$$\therefore f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx.$$



(b) $$b_n = \frac{2}{\pi}\int_0^\pi x^2 \sin nx\,dx = \frac{4(-1)^n - 4}{n^3\pi} - \frac{2\pi(-1)^n}{n}$$

$$\therefore f(x) = 2\sum_{n=1}^{\infty}\left[\frac{2(-1)^n - 2}{n^3\pi} - \frac{\pi(-1)^n}{n}\right]\sin nx.$$



(c) Extend the definition of f to $f(x) = \begin{cases} (x+\pi)^2, & -\pi < x < 0 \\ x^2, & 0 \le x < \pi \end{cases}$

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^0 (x+\pi)^2\,dx + \int_0^\pi x^2\,dx\right] = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^0 (x+\pi)^2 \cos nx\,dx + \int_0^\pi x^2 \cos nx\,dx\right] = \frac{1}{n^2}$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^0 (x+\pi)^2 \sin nx\,dx + \int_0^\pi x^2 \sin nx\,dx\right] = -\frac{\pi}{n}$$

$$\therefore f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{1}{n^2}\cos nx - \frac{\pi}{n}\sin nx\right].$$