TW364 Lecture 3

A function f is periodic with period T if $f(x+T) = f(x)$.

Example: $\sin x$ is periodic with period 4π,
since $\sin(x+4\pi) = \sin x$

The smallest $T > 0$ for which $f(x+T) = f(x)$ is called the fundamental period of f.

Example: the fundamental period of $\sin x$ is $T = 2\pi$.

Fourier series: periodic extension

The fundamental period of both $\cos \frac{n\pi x}{p}$ and $\sin \frac{n\pi x}{p}$ is $\frac{2p}{n}$.

Since a positive integer multiple of a period is also a period, all of the Fourier basis functions have the period $\boxed{2p}$ in common, and this is the smallest period common to them.

So the Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{p} + b_n \sin \frac{n\pi x}{p} \right]$$

is periodic with fundamental period $2p$.

The Fourier series does not only represent $f(x)$ on $[-p, p]$, but also gives the periodic extension of f outside $[-p, p]$.

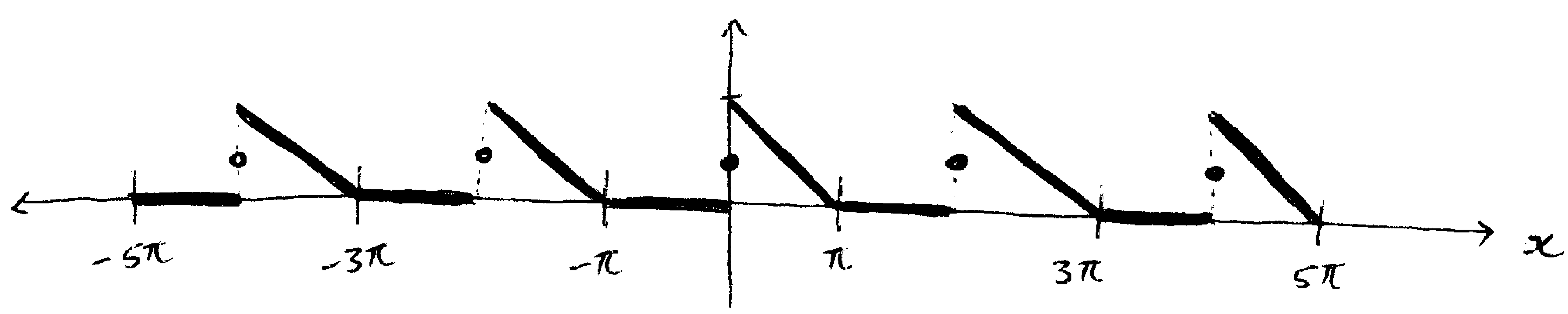
Example

We saw that the Fourier series of $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \pi - x, & 0 \leq x < \pi \end{cases}$

is given by

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left[\frac{1-(-1)^n}{n^2 \pi} \cos nx + \frac{1}{n} \sin nx \right].$$

The expression on the RHS converges to the periodic extension of f, with period 2π:



At $x = 0, \pm 2\pi, \pm 4\pi, \ldots$ the series converges to

$$\frac{1}{2}\left[f(0^+) + f(0^-)\right] = \frac{1}{2}\left[\pi + 0\right] = \frac{\pi}{2}$$ (the dots)

At $x = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$ the series converges to

$$\frac{1}{2}\left[f(-\pi^+) + f(\pi^-)\right] = \frac{1}{2}\left[0 + 0\right] = 0$$

Partial sum approximations

The partial sum

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N-1} \left[a_n \cos \frac{n\pi x}{p} + b_n \sin \frac{n\pi x}{p} \right]$$

approximates $f(x)$, and the approximation improves for larger N.

Example

In Tutorial 1, Problem 3(b) we saw that the Fourier series of

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x^2, & 0 \le x < \pi \end{cases}$$

is $$f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \left[\frac{2(-1)^n}{n^2} \cos nx + \left(\frac{2(-1)^n - 2}{n^3 \pi} - \frac{(-1)^n \pi}{n} \right) \sin nx \right]$$

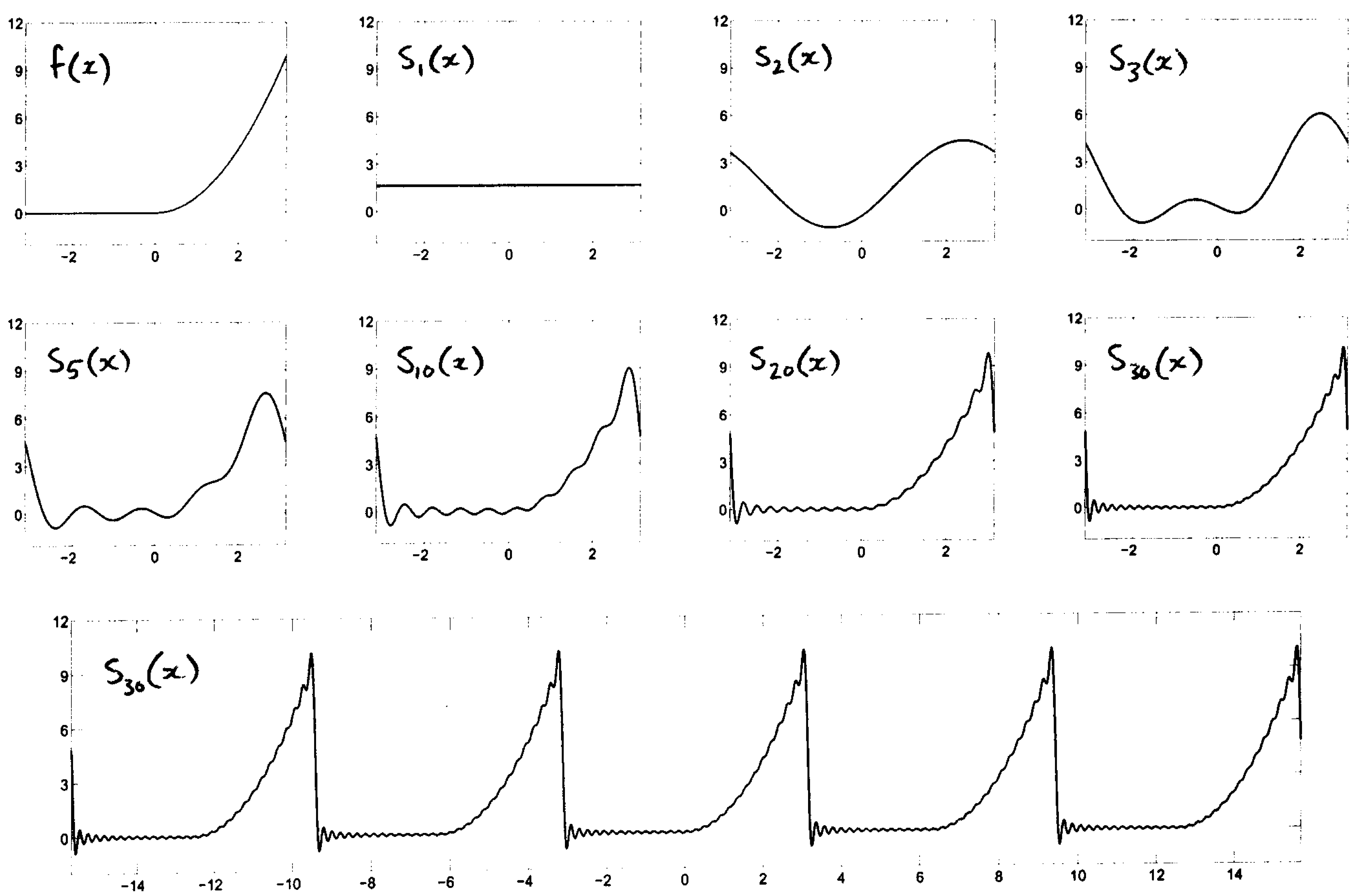
Partial sum approximations of f:

$$S_1(x) = \frac{\pi^2}{6}$$

$$S_2(x) = \frac{\pi^2}{6} - 2\cos x + \left(\pi - \frac{4}{\pi}\right) \sin x$$

$$S_3(x) = \frac{\pi^2}{6} - 2\cos x + \left(\pi - \frac{4}{\pi}\right) \sin x + \frac{1}{2} \cos 2x - \frac{\pi}{2} \sin 2x$$

$\vdots$



Fourier series in complex form

From Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$,

we get
$$\cos\theta = \tfrac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$
$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) = -\frac{i}{2}\left(e^{i\theta} - e^{-i\theta}\right)$$

Substitute these into the Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{p} + b_n \sin\frac{n\pi x}{p}\right]$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\tfrac{1}{2}a_n\left(e^{in\pi x/p} + e^{-in\pi x/p}\right) - \tfrac{i}{2}b_n\left(e^{in\pi x/p} - e^{-in\pi x/p}\right)\right]$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\left(\frac{a_n - ib_n}{2}\right)e^{in\pi x/p} + \left(\frac{a_n + ib_n}{2}\right)e^{-in\pi x/p}\right]$$
$$= \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/p}$$

where $c_0 = \frac{a_0}{2}$, $c_n = \frac{a_n - ib_n}{2}$, $c_{-n} = \frac{a_n + ib_n}{2}$, $n = 1, 2, 3, \ldots$

From the definitions of a_0, a_n and b_n

$$c_0 = \frac{1}{2p}\int_{-p}^{p} f(x)\,dx$$

$$c_n = \frac{1}{2p}\int_{-p}^{p} f(x)\left[\cos\frac{n\pi x}{p} - i\sin\frac{n\pi x}{p}\right]dx$$
$$= \frac{1}{2p}\int_{-p}^{p} f(x)e^{-in\pi x/p}\,dx, \quad n = 1, 2, 3, \ldots$$

$$c_{-n} = \frac{1}{2p}\int_{-p}^{p} f(x)\left[\cos\frac{n\pi x}{p} + i\sin\frac{n\pi x}{p}\right]dx$$

$$= \frac{1}{2p}\int_{-p}^{p} f(x)\, e^{in\pi x/p}\, dx, \quad n = 1, 2, 3, \ldots$$

$$\therefore \ c_n = \frac{1}{2p}\int_{-p}^{p} f(x)\, e^{-in\pi x/p}\, dx, \quad n = -1, -2, -3, \ldots$$

To summarize, the complex form of the Fourier series of a function f on $[-p, p]$ is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/p}$$

where $c_n = \frac{1}{2p}\int_{-p}^{p} f(x)\, e^{-in\pi x/p}\, dx, \quad n = 0, \pm 1, \pm 2, \ldots$