TW364 Lecture 2

Orthogonal vector expansion

Consider 3 mutually orthogonal vectors $\underline{v}_1, \underline{v}_2, \underline{v}_3$ in $\mathbb{R}^3$.
They form a basis for $\mathbb{R}^3$, such that any given vector $\underline{u} \in \mathbb{R}^3$ can be written as

$$\underline{u} = c_1 \underline{v}_1 + c_2 \underline{v}_2 + c_3 \underline{v}_3 .$$

How do we determine the coefficients c_1, c_2, c_3?

Take the inner product with $\underline{v}_1$ on both sides:

$$(\underline{u}, \underline{v}_1) = c_1 (\underline{v}_1, \underline{v}_1) + c_2 (\underline{v}_2, \underline{v}_1) + c_3 (\underline{v}_3, \underline{v}_1)$$
$$= c_1 \|\underline{v}_1\|^2 + c_2 \cdot 0 + c_3 \cdot 0$$

Hence $c_1 = \dfrac{(\underline{u}, \underline{v}_1)}{\|\underline{v}_1\|^2}$

Similarly, $c_2 = \dfrac{(\underline{u}, \underline{v}_2)}{\|\underline{v}_2\|^2}$ and $c_3 = \dfrac{(\underline{u}, \underline{v}_3)}{\|\underline{v}_3\|^2}$.

Orthogonal series expansion of functions

Consider a set $\{\phi_0(x), \phi_1(x), \phi_2(x), \dots\}$ orthogonal on $[a,b]$, and suppose $f(x)$ is defined on $[a,b]$.

We're looking for coefficients $c_0, c_1, c_2, \dots$ such that

$$f(x) = c_0 \phi_0(x) + c_1 \phi_1(x) + c_2 \phi_2(x) + \dots \qquad \boxed{*}$$

Multiply $\boxed{*}$ by $\phi_n(x)$ and integrate :

$$\int_a^b f(x)\,\phi_n(x)\,dx$$

$$= c_0 \int_a^b \phi_0(x)\,\phi_n(x)\,dx + c_1 \int_a^b \phi_1(x)\,\phi_n(x)\,dx + \cdots$$

$$\therefore \ (f, \phi_n) = c_0(\phi_0, \phi_n) + c_1(\phi_1, \phi_n) + c_2(\phi_2, \phi_n) + \cdots$$

$$= c_n \|\phi_n\|^2$$

Therefore $\displaystyle f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$ where $\displaystyle c_n = \frac{(f, \phi_n)}{\|\phi_n(x)\|^2}$

<u>Complete sets</u>

Note that to expand f in a series of orthogonal functions $\{\phi_0, \phi_1, \cdots\}$, it is necessary that f not be orthogonal to each ϕ_n.

To avoid this problem, we shall assume that an orthogonal set is always <u>complete</u>. That is, the only function that is orthogonal to each member of the set is the zero function.

<u>Convergence</u>

Does the expansion in $\boxed{*}$ actually exist, and does it converge to f ?

We'll address this issue later on, for specific $\phi_0, \phi_1, \cdots$.

[Short answer : usually, yes...]

11.2 FOURIER SERIES

Consider the set of functions

$$\left\{ 1, \cos\frac{\pi x}{p}, \cos\frac{2\pi x}{p}, \cos\frac{3\pi x}{p}, \dots, \sin\frac{\pi x}{p}, \sin\frac{2\pi x}{p}, \sin\frac{3\pi x}{p}, \dots \right\}$$

which is orthogonal on the interval $[-p, p]$.

(Tutorial 1, Problem 2 asks you to prove orthogonality for $p = \pi$. Extending the proof for a general p is straightforward.)

Suppose f is defined on $[-p, p]$, and can be expanded as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\frac{n\pi x}{p} + b_n \sin\frac{n\pi x}{p} \right]. \quad \boxed{**}$$

Using the result in the middle of the previous page, we find

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\, dx$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x) \cos\frac{n\pi x}{p}\, dx$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x) \sin\frac{n\pi x}{p}\, dx$$

With these coefficients, $\boxed{**}$ is known as the <u>Fourier series</u> of the function f on the interval $[-p, p]$.

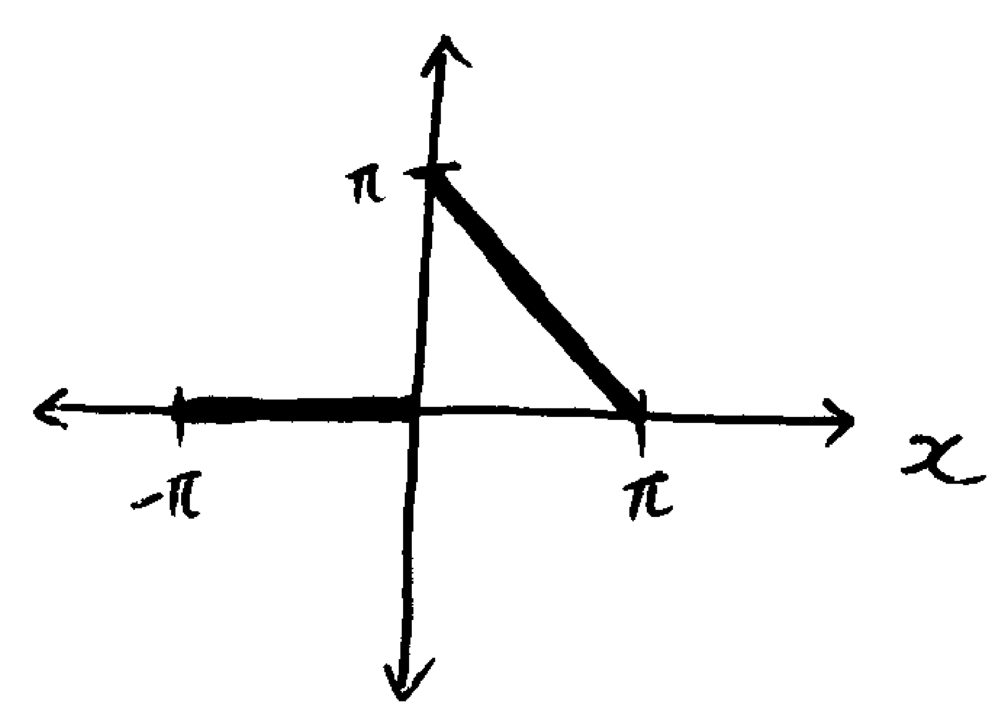
Conditions for convergence

Theorem 11.2.1, p. 428:

If f and f' are piecewise continuous on $[-p, p]$, the Fourier series of f converges to $f(x)$ for all point continuities. At a point discontinuity x, the Fourier series converges to the average of the right- and left-hand limits of f at x, that is $\frac{1}{2}\left(f(x^+) + f(x^-)\right)$.

Example

Find the Fourier series of $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases}$



Note that both $f(x)$ and $f'(x) = \begin{cases} 0, & -\pi < x < 0 \\ -1, & 0 \le x < \pi \end{cases}$ are piecewise continuous on $[-\pi, \pi]$.

Fourier series coefficients $(p = \pi)$:

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} (\pi - x)\,dx\right] \\ &= \frac{1}{\pi}\left[\pi x - \frac{1}{2}x^2\right]_0^{\pi} \\ &= \frac{\pi}{2} \end{aligned}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \, dx + \int_{0}^{\pi} (\pi - x)\cos nx \, dx\right]$$

$$= \frac{1}{\pi}\left[\ 0 \ + \frac{1}{n}(\pi - x)\sin nx \Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi} \sin nx \, dx\right]$$

$$= \frac{1}{\pi}\left[\ 0 \ + \ 0 \ - \frac{1}{n^2}\cos nx \Big|_0^{\pi}\ \right]$$

$$= \frac{1-(-1)^n}{n^2\pi} \qquad \left[\text{Note : } \cos n\pi = (-1)^n\right]$$

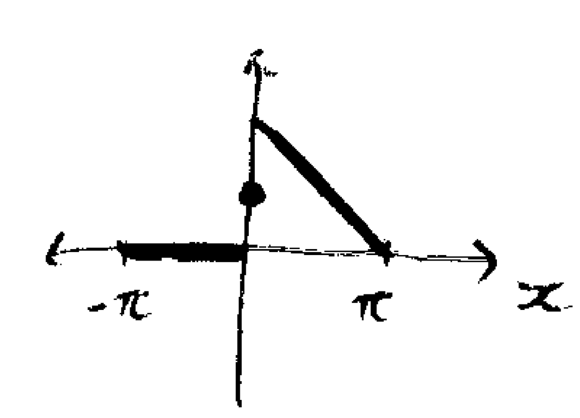
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx = \frac{1}{n} \qquad \left[\text{check it yourself!}\right]$$

Therefore

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2\pi}\cos nx + \frac{1}{n}\sin nx\right].$$

Note : $x = 0$ is a point discontinuity of f. At $x = 0$ the above Fourier series will converge to

$$\frac{1}{2}\left(f(0^+) + f(0^-)\right) = \frac{1}{2}(\pi + 0) = \frac{\pi}{2}$$



Also note that even though the formula for a_n appears to reduce to a_0 when $n = 0$, this is not the case <u>after</u> the integrals are evaluated (as the example above shows).