TW364 Lecture 1

II.1 ORTHOGONAL FUNCTIONS

Inner product of vectors

$$(\underline{u}, \underline{v}) = \underline{u} \cdot \underline{v} = \sum_{i=1}^{n} u_i v_i \quad , \quad \underline{u}, \underline{v} \in \mathbb{R}^n$$

Two vectors $\underline{u}$ and $\underline{v}$ are orthogonal if $(\underline{u}, \underline{v}) = 0$.

The norm of $\underline{u}$ is $\| \underline{u} \| = \sqrt{(\underline{u}, \underline{u})}$.

Inner product of functions

The inner product of two functions f_1 and f_2 on an interval $[a,b]$ is defined as

$$(f_1, f_2) = \int_a^b f_1(x)\, f_2(x)\, dx .$$

Some properties (easily proved) :

* $(f_1, f_2) = (f_2, f_1)$
* $(kf_1, f_2) = k(f_1, f_2)$ with k a constant
* $(f, f) = 0$ if $f(x)$ is the zero function

  $(f, f) > 0$ if $f(x)$ is not the zero function

Two functions f_1 and f_2 are orthogonal on $[a,b]$ if

$$(f_1, f_2) = 0$$

Examples

a) $f(x) = x^2$ and $g(x) = x^3$ are orthogonal on $[-1,1]$, since

$$(f,g) = \int_{-1}^{1} x^2 \cdot x^3 \, dx = \frac{1}{6} x^6 \Big|_{-1}^{1} = 0.$$

b) $f(x) = x^2$ and $h(x) = x^4$ are not orthogonal on $[-1,1]$, since

$$(f,h) = \int_{-1}^{1} x^2 \cdot x^4 \, dx = \frac{1}{7} x^7 \Big|_{-1}^{1} = \frac{2}{7} \neq 0.$$

Orthogonal set

An infinite set of functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \dots\}$ is orthogonal on $[a,b]$ if $(\phi_m, \phi_n) = 0$ for $m \neq n$.

Orthonormal set

The norm of a function on $[a,b]$ is $\|f(x)\| = \sqrt{(f,f)} = \sqrt{\int_a^b f^2(x)\,dx}$

An orthogonal set of functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \dots\}$ with the additional property that $\|\phi_n(x)\| = 1$ for $n = 0, 1, 2, \dots$ is said to be orthonormal.

Example

Consider the set $\{1, \cos x, \cos 2x, \dots\}$ on $[-\pi,\pi]$.

Let $\phi_0(x) = 1$, $\phi_1(x) = \cos x$, $\phi_2(x) = \cos 2x$, etc.

Now, $$(\phi_0, \phi_n) = \int_{-\pi}^{\pi} \cos nx \, dx = \frac{1}{n} \sin nx \Big|_{-\pi}^{\pi} = 0, \quad n \neq 0$$

$$(\phi_m, \phi_n) = \int_{-\pi}^{\pi} \cos mx \cos nx \, dx$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \left[\cos(m+n)x + \cos(m-n)x \right] dx$$

$$= \frac{1}{2} \left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n} \right]_{-\pi}^{\pi}$$

$$= 0, \quad m \neq n$$

$\therefore$ The set $\{1, \cos x, \cos 2x, \dots\}$ is orthogonal on $[-\pi,\pi]$.

Normalization

For the example above,

$$\| \phi_0(x) \| = \sqrt{\int_{-\pi}^{\pi} dx} = \sqrt{2\pi}$$

$$\| \phi_n(x) \| = \sqrt{\int_{-\pi}^{\pi} \cos^2 nx \, dx} = \sqrt{\pi}, \quad n \neq 0$$

$\therefore$ The set $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \dots \right\}$ is orthonormal on $[-\pi,\pi]$.