Theorem 12

There are exactly five platonic solids.

Proof: Let P be a platonic solid and let $G(P)$ be its associated plane graph. Let p, q and r be the number of vertices, edges and regions of $G(P)$ respectively.

Let m be the degree of each region and let n be the degree of each vertex.

By adding the degrees of the regions over all regions, we get $2q = mr$, and by adding the degrees of the vertices over all vertices, we get $2q = np$.

It follows from Euler's theorem that

$$0 < 2 = p - q + r = \frac{2q}{n} - q + \frac{2q}{m}$$

$$= q\left[\frac{2m - mn + 2n}{mn}\right].$$

Therefore $2m - mn + 2n > 0$,

hence $(m-2)(n-2) < 4$.

Since $m, n \geq 3$ it follows that this inequality has the five solutions

1. $m = 3$ and $n = 3$. In this case $G(P)$ is a 3-regular triangulation. By Theorem 10, $3p_3 = 12$, hence $p = p_3 = 4$, and so $r = \frac{np}{m} = 4$. Therefore P is a platonic solid constructed from 4 triangles, it has 4 vertices, each of degree 3. This is the tetrahedron.

2. $m = 4$ and $n = 3$. Here $G(P)$ is 3-regular with all its regions of degree 4. By Theorem 11, $2r_4 = 12$, hence $r = r_4 = 6$, and so $p = \frac{mr}{n} = 8$. Therefore P is a platonic solid constructed from 6 squares, it has 8 vertices, each of degree 3. This is the cube.

3. $m = 3$ and $n = 4$. $G(P)$ is a 4-regular triangulation. By Theorem 10, $2p_4 = 12$, hence $p = p_4 = 6$, and so $r = \frac{np}{m} = 8$. Therefore P is a platonic solid constructed from 8 triangles, it has 6 vertices, each of degree 4.

4. $m = 5$ and $n = 3$. $G(P)$ is a 3-regular plane graph with all its regions of degree 5. By Theorem 11, $r = r_5 = 12$, and so $p = \frac{mr}{n} = 20$. Therefore P is constructed from 12 pentagons, it has 20 vertices of degree 3. This is the dodecahedron.

5. $m = 3$ and $n = 5$. $G(P)$ is a 5-regular triangulation. By Theorem 10, $p = p_5 = 12$, and so $r = \frac{np}{m} = 20$. Therefore P is constructed from 20 triangles, it has 12 vertices of degree 5. This is the icosahedron.