TW 314 Lecture 14 (We 1 Mar. 2017)

MAXIMAL PLANAR GRAPHS

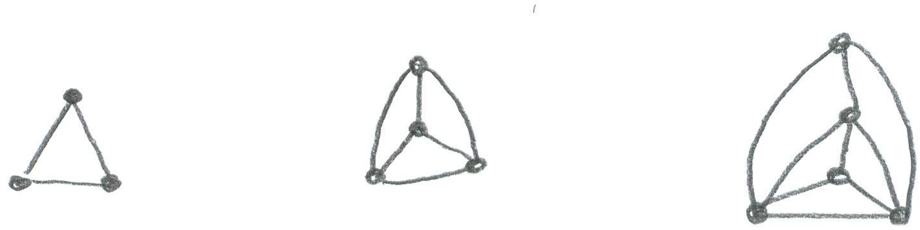
Definition A planar graph G is a maximal planar graph if, for any pair of non-adjacent vertices u and v of G, $G + uv$ is non-planar.

Examples (of maximal planar graphs)

K_1 K_2 K_3 K_4

In any embedding of a maximal planar graph with more than two vertices, the boundary of every region must be a triangle (a 3-cycle). For this reason such graphs are often called triangulations.

Examples (of triangulations)



Theorem 8 If G is a maximal planar graph of order $p \geq 3$ and size q, then $q = 3p - 6$.

Proof: Embed G in the plane, and say there are r regions. Since $p \geq 3$, the boundary of every region is a triangle and each edge is on the boundary of exactly two regions. Let N be the sum, over all the regions, of number of edges on the boundary of a region. Then $N = 3r$ since the boundary of each region is a triangle, and $N = 2q$ since each edge is counted twice. Therefore $3r = 2q$. It now follows from Euler's theorem that $q = 3p - 6$. ∎

Corollary 9 If G is a planar graph of order $p \geq 3$ and size q, then $q \leq 3p - 6$.

Proof: Let H be a maximal planar graph of order p that contains G as a subgraph. Then H has size $q' \geq q$ and $q' = 3p - 6$. Therefore $q \leq 3p - 6$. ∎

Theorem 10 Let G be a maximal planar graph of order $p \geq 4$, and let p_i denote the number of vertices of degree i in G.

Then $3p_3 + 2p_4 + p_5 = 12 + p_7 + 2p_8 + \cdots + (i-6)p_i + \cdots$

Proof: Since $p \geq 4$, the are no vertices of degree 0, 1 or 2. Therefore $p = \sum_{i=3}^{\infty} p_i$. Let the size of G be q. Then $2q = \sum_{i=3}^{\infty} i p_i$, and since G is maximal planar, $q = 3p - 6$.

Therefore $\sum_{i=3}^{\infty} i p_i = \sum_{i=3}^{\infty} 6p_i - 12$ or $\sum_{i=3}^{\infty} (6-i)p_i = 12$,

and hence $3p_3 + 2p_4 + p_5 = 12 + \sum_{i=7}^{\infty} (i-6)p_i$ ∎

Examples K_4 has $p_3 = 4$ (all other $p_i = 0$)

and $3 \times 4 = 12$

The maximal planar graph G

has $r_3 = 2$, $r_4 = 3$ (all other $p_i = 0$)

and $3 \times 2 + 2 \times 3 = 12$.

G: