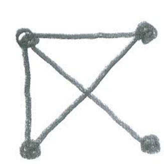
GRAPHICAL SEQUENCES

Definition: A sequence $d_1, d_2, \ldots, d_n$ of non-negative integers is a **degree sequence** of a graph G if G has order n and $d_1, d_2, \ldots, d_n$ are the degrees of the vertices of G.

A **graphical sequence** is a sequence of integers for which there exists a graph G such that the sequence is the degree sequence of G.

Examples: The (unlabelled) graph G has degree sequence $3, 2, 2, 1$ and therefore $3, 2, 2, 1$ is a graphical sequence.

G:



The sequence $3, 3, 2, 1$ is not graphical.

Why?

A characterization of graphical sequences is given by the following result due to Havel (1955) and Hakimi (1962).

__Theorem 3__ (The Havel-Hakimi Theorem)

A sequence $s: d_1, d_2, \ldots, d_n$ of non-negative integers with $d_1 \geq d_2 \geq \ldots \geq d_n$ ($n \geq 2$ and $d_1 \geq 1$) is graphical if and only if the sequence

$s_1: d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, d_{d_1+3}, \ldots, d_n$

is graphical.

__Proof:__ Assume first that s_1 is graphical and let G_1 be a graph of order $n-1$ with degree sequence s_1. Let the vertices of G_1 be $v_2, v_3, \ldots, v_n$ with $\deg_{G_1}(v_i) = d_i - 1$ if $2 \leq i \leq d_1+1$

and $\deg_{G_1}(v_i) = d_i$ if $d_1+2 \leq i \leq n$.

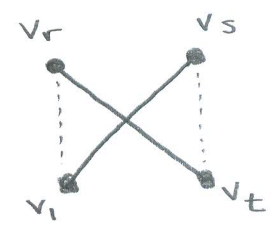
Construct the graph G by adding a new vertex v_1 to G_1 as well as the d_1 edges vv_i for $i = 2, 3, \ldots, d_1+1$. Then G has degree sequence s with $\deg_G(v_i) = d_i$ for $i = 1, 2, \ldots, n$.

For the converse, assume that s is graphical. Of all the graphs of order n with degree sequence s, let G be one with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $\deg_G(v_i) = d_i$ for $i = 1, 2, \ldots, n$ and such that the sum of the degrees of the vertices adjacent to v_1 is maximal.

We now show that v_1 is adjacent to $v_2, v_3, \ldots, v_{d_1+1}$ from which it follows that the graph $G - v_1$ has degree sequence s_1.

Suppose to the contrary that there are two vertices v_r and v_s with $d_r > d_s$ and such that v_1 is adjacent to v_s but not to v_r.

G:



Since $d_r > d_s$, there is a vertex v_t such that v_t is adjacent to v_r but not to v_s. Construct the graph G' from G by removing the edges $v_1 v_s$ and $v_r v_t$ and adding the edges $v_1 v_r$ and $v_s v_t$.

G':