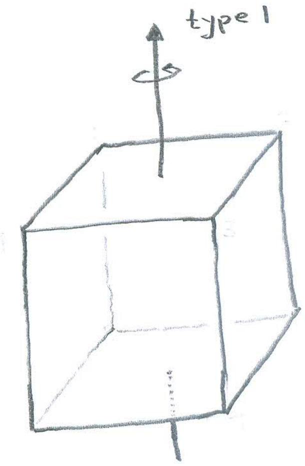
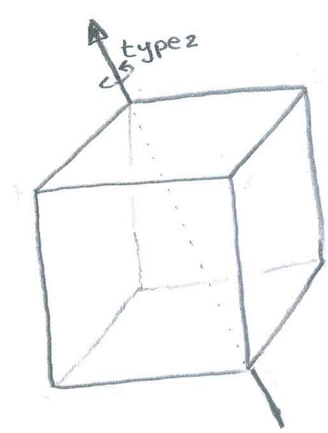
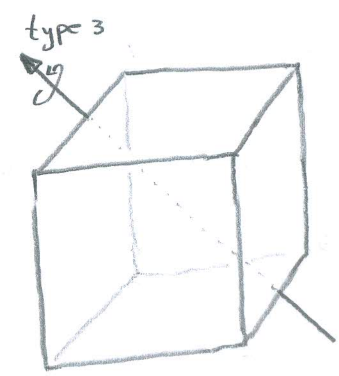
Problem 4 In how many ways can the corners of a cube be coloured in three colours?

Solution: Let the group G of the 24 symmetries of the cube act on the set X of the 3^8 colourings obtained by colouring each of the eight corners of the cube in any of three colours — the cube is for now considered fixed in space, no transformations are yet considered.

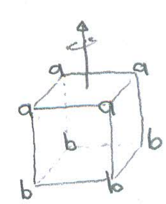




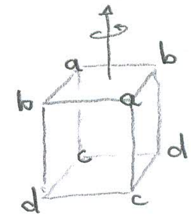


The number of different colourings is given by the number of orbits of X under the action of G, which is $\frac{1}{|G|}\sum_{g\in G}|X_g|$.

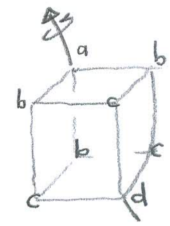
$g \in G$	$\lvert X_g \rvert$
id	3^8
6 rotations through 90° around axes of type 1	6×3^2



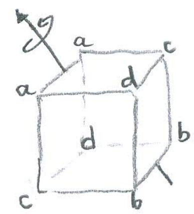
3 rotations through 180° around axes of type 1 $\Big\}$ --- 3×3^4



8 rotations through 120° around axes of type 2 $\Big\}$ --- 8×3^4



6 rotations through 180° around axes of type 3 $\Big\}$ --- 6×3^4



$$\sum_{g \in G} |X_g| = 3^8 + 6 \times 3^2 + 3 \times 3^4 + 8 \times 3^4 + 6 \times 3^4$$

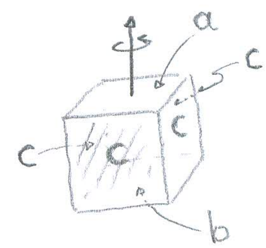
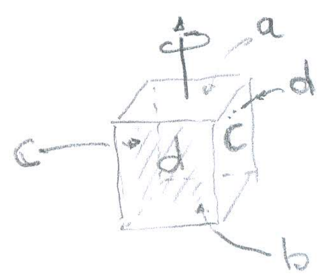
$$= 7992.$$

Therefore the number of different colourings is

$$\frac{1}{|G|} \sum_{g \in G} |X_g| = \frac{7992}{24} = 333$$

Problem 5 In how many ways can the faces of a cube be coloured in three colours?

Solution: Let the group G of symmetries of the cube act on the set X of the 3^6 colourings of the 6 faces of the cube — before any transformations are considered. We need to find $\frac{1}{|G|} \sum_{g \in G} |X_g|$.

$g \in G$	$\lvert X_g \rvert$	
id	3^6	
6 rotations through 90° around axes of type 1	6×3^3	
3 rotations through 180° around axes of type 1	3×3^4	
8 rotations through 120° around axes of type 2	8×3^2