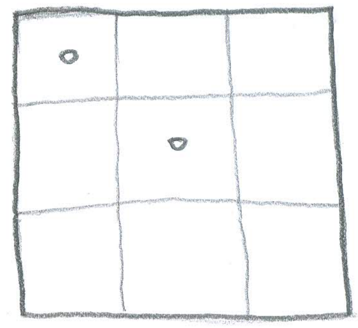
TW 314 Lecture 7 (Mo 13 Feb 2017)

Problem 1 (Identity Cards)



A square card, 3×3 grid, two holes punched in two of the nine small squares.

How many such identity cards are there?

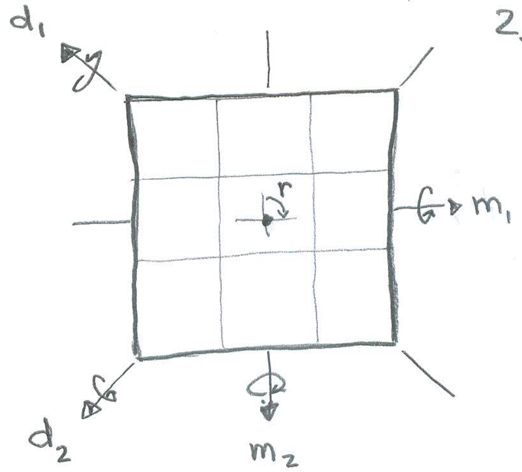
Solution: Let G be the group of 8 symmetric transformations of the square, and let G act on the set X of all $\binom{9}{2} = 36$ configurations (before any transformations of the square are made) obtained by punching holes in two of the nine small squares. Then,

the # of different identity cards

= the # of orbits of X under the action of G

$$= \frac{1}{|G|} \sum_{g \in G} |X_g| .$$ We know that $|G| = 8$.

We now need to find $\sum_{g \in G} |X_g|$.



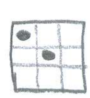
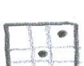
$G = \{id, r, r^2, r^3, m_1, m_2, d_1, d_2\}$

where r is the clockwise rotation through $90°$, so r^2 is the clockwise rotation through $180°$, and r^3 through $270°$.

$g \in G$		$\lvert X_g \rvert$
id	———	36
r	———	0
r^2	———	4
r^3	———	0
m_1	———	$3 + \binom{3}{2} = 6$
m_2	———	$3 + \binom{3}{2} = 6$
d_1	———	$3 + \binom{3}{2} = 6$
d_2	———	$3 + \binom{3}{2} = 6$

$\sum_{g \in G} \lvert X_g \rvert = 64$, and therefore the

$\#$ of orbits $= \dfrac{64}{8} = 8.$

(Note that, if x is the configuration , then its orbit is $Gx = \{$ , , ,  $\}$ and its stabilizer is $G_x = \{id, d_1\}$ and $\lvert Gx \rvert \lvert G_x \rvert = 4 \times 2 = 8 = \lvert G \rvert$.)

Problem 2 (Necklaces)

Sixteen beads on a loop of string — 3 black and 13 white.

How many such necklaces are there?

Solution: Let G be the group of symmetric transformations of a regular polygon with 16 sides. Let X be the set of all configurations obtained by choosing three of the corners of the polygon to be black, and the others white.

Then # of necklaces

= # of orbits of X under the actions of G

$$= \frac{1}{|G|} \sum_{g \in G} |X_g|.$$

$g \in G$	$\lvert X_g \rvert$
id	$\lvert X \rvert = \binom{16}{3} = 560$
15 rotations through $\frac{360^\circ}{16} \times n$ for $n = 1, 2, \ldots, 15$	0
8 reflections through corners	14
8 reflections through sides	0