The size of an orbit

Theorem 14.3 If G is a group of permutations of the set X and $x \in X$, then

$$\boxed{|Gx||G_x| = |G|}$$

Proof: Let $S = \{(g, y) : g(x) = y\}$. Now count the number of elements of S in two ways.

For each $g \in G$, the number of pairs (g, y) for which $g(x) = y$ is 1.
Therefore $|S| = \sum_{g \in G} 1 = |G|$.

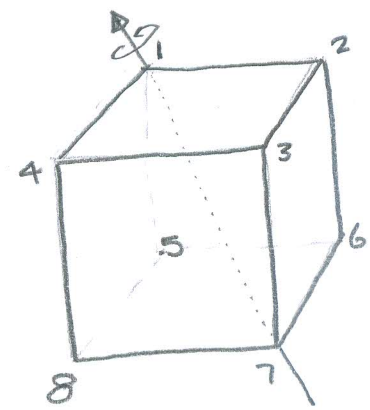
For each $y \in X$, the number of pairs (g, y) for which $g(x) = y$ is $|G(x \to y)|$.
Therefore

$$\begin{aligned}
|S| &= \sum_{y \in X} |G(x \to y)| \\
&= \sum_{\substack{y \in X \\ y \in Gx}} |G(x \to y)| + \sum_{\substack{y \in X \\ y \notin Gx}} |G(x \to y)| \\
&= \sum_{y \in Gx} |G_x| + 0 = |Gx||G_x|.
\end{aligned}$$

It follows that $|Gx||G_x| = |G|$. ∎

Example: How many symmetries does the cube have?

Solution: Let G be the group of symmetries of the cube, and let G act on the set $X = \{1,2,3,4,5,6,7,8\}$ of corners of the cube.



Then $G_1 = \{ id, (254)(368), (245)(386) \}$,
so $|G_1| = 3$. It is not difficult to see that 1 can be taken to any element of X by a permutation of G. For example, $(1265)(4378)$ takes 1 to 2, $(16)(25)(47)(38)$ takes 1 to 6, and $(17)(53)(48)(26)$ takes 1 to 7.
Therefore $G1 = X$, and so $|G1| = 8$. It follows that $|G| = |G1||G_1| = 3 \times 8 = 24$ — so the cube has 24 symmetries.

(Note that, since $G1 = X$, there is only one orbit, and therefore $G1 = G2 = \cdots = G8$.)

The number of orbits

Theorem 14.4 The number of orbits of a group G acting on a set X is given by

$$\boxed{\frac{1}{|G|}\sum_{g\in G}|X_g|}$$

where $X_g = \{x \in X : g(x) = x\}$.

Proof: Let $E = \{(g,x) : g(x) = x\}$. Now count the number of elements of E in two ways.

For each $g \in G$, the number of pairs (g,x) for which $g(x) = x$ is $|X_g|$.
Therefore $|E| = \sum_{g\in G}|X_g|$.

For each $x \in X$, the number of pairs (g,x) for which $g(x) = x$ is $|G_x|$.

Therefore $|E| = \sum_{x\in X}|G_x| = \sum_{x\in X}\frac{|G|}{|Gx|}$.

It follows that

$$\frac{1}{|G|}\sum_{g\in G}|X_g| = \sum_{x\in X}\frac{1}{|Gx|}.$$