TW 314 Lecture 5 (Wo 8 Feb. 2017)

Theorem 14.2 If G is a group of permutations of X and $h \in G(x \to y)$, then

$$\boxed{G(x \to y) = hG_x}$$

which is the left coset of G_x with respect to h.

Proof:

$$\alpha \in hG_x \Rightarrow \alpha = h\beta \text{ for some } \beta \in G_x$$
$$\Rightarrow \alpha(x) = h\beta(x) = h(x) = y$$
$$\Rightarrow \alpha \in G(x \to y)$$

Therefore $hG_x \subseteq G(x \to y)$.

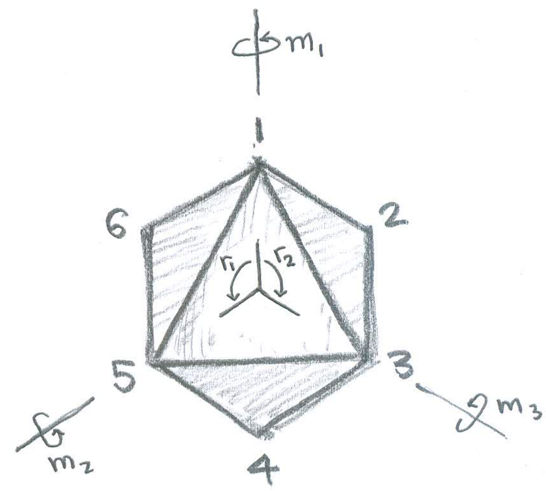
$$\gamma \in G(x \to y) \Rightarrow \gamma(x) = y$$
$$\Rightarrow h^{-1}\gamma(x) = h^{-1}(y) = x$$
$$\Rightarrow h^{-1}\gamma \in G_x$$
$$\Rightarrow \gamma \in hG_x$$

Therefore $G(x \to y) \subseteq G_x$.

It follows that $G(x \to y) = h(G_x)$. ∎

Use Theorem 14.2 to find the cardinality of $G(x \to y)$ as follows:

$$\boxed{\begin{aligned} |G(x\to y)| &= |G_x| \quad \text{if } y \in Gx \\ &= 0 \quad \text{if } y \notin Gx \end{aligned}}$$



__Example:__ A regular two-dimensional figure constructed on a hexagon with corners 1, 2, 3, 4, 5, 6.

It has the group of symmetries $G = \{id, r_1, r_2, m_1, m_2, m_3\}$ which acts on the set $X = \{1, 2, 3, 4, 5, 6\}$ of the six corners of the hexagon.

Explicitly, $r_1 = (1\,3\,5)(2\,4\,6)$ $\quad r_2 = (1\,5\,3)(2\,6\,4)$

$m_1 = (2\,6)(3\,5)$ $\quad m_2 = (1\,3)(4\,6)$ $\quad m_3 = (2\,4)(1\,5)$.

The orbits are

$$G1 = G3 = G5 = \{1, 3, 5\}$$

$$G2 = G4 = G6 = \{2, 4, 6\}.$$

The stabilizers are

$$G_1 = G_4 = \{id, m_1\}$$
$$G_2 = G_5 = \{id, m_2\}$$
$$G_3 = G_6 = \{id, m_3\}$$

The three left cosets of G_1 are

$$G_1 = \{id, m_1\}$$
$$m_2 G_1 = \{m_2, m_2 m_1\} = \{m_2, r_1\} = G(1 \to 3)$$
$$m_3 G_1 = \{m_3, m_3 m_1\} = \{m_3, r_2\} = G(1 \to 5).$$

It may be helpful to depict the action of $G = \{id, r_1, r_2, m_1, m_2, m_3\}$ on $X = \{1, 2, 3, 4, 5, 6\}$ as follows: