Voorl. en Van / *Init. and Surname:* MEMO
Studentenommer/ *Student number:*

TW/AM 214 TOETS 1 / *TEST 1* 2017

TOTAAL: 70 punte, TYDSUUR: 90 min. *TOTAL: 70 marks, DURATION: 90 mins.*

1 Gee voorbeelde van 3×2 stelsels van vergelykings met
(a) slegs een unieke oplossing,

Give examples of 3×2 systems of equations with [6]
(a) only one unique solution,

TYPICAL EXAMPLE

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

(b) geen oplossing, *(b) no solution,*

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

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(c) oneindig veel oplossings. *(b) infinitely many solutions.*

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

2 Laat A, B en X $(n \times n)$-matrikse wees en I is die $(n \times n)$-identiteitsmatriks. Neem aan dat waar ookal 'n inverse van 'n matriks benodig word, dit wel bestaan. Maak in X die onderwerp van die formule in onderstaande vergelyking, en vereenvoudig die resultaat so ver as moontlik.

Let A, B and X be $(n \times n)$ matrices and let I be the $(n \times n)$ identity matrix. Assume that whenever an inverse of a matrix is needed, it exists. Make X the subject of the formula in the equation below, and simplify the result as far as possible. [6]

$$(A^{-1} + I)(X + I) - B^T = I + A^{-1}B^T + A$$

Premult. by A: $(I+A)(X+I) - AB^T = A + B^T + A^2$

$$(I+A)(X+I) = A + A^2 + AB^T + B^T$$
$$= (I+A)A + (I+A)B^T$$
$$= (I+A)(A+B^T)$$

Premult. by $(I+A)^{-1}$:

$$X + I = A + B^T$$
$$X = A + B^T - I$$

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$\boxed{X = A + B^T - I}$

2. Correct versions, but not fully simplified

$X = B^T + (A^{-1} + I)^{-1}(A - A^{-1})$ 4/6

$X = (A^{-1} + I)^{-1}(A^{-1}B^T + A + B^T - A^{-1})$ 4/6

$X = B^T + (A + I)^{-1}(A^2 - I)$ 4/6

$X = (I + A)^{-1}(-I + AB^T + B^T + A^2)$ 4/6

$X = (A^{-1} + I)^{-1}(I + A^{-1}B^T + A + B^T) - I$ 4/6

$X = (A^{-1} + I)^{-1}(A + I) + B^T - I$ 4/6

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3 'n Vektorruimte gehoorsaam twee reëls: (1) geslotenheid onder optelling, en (2) geslotenheid onder skalaarvemenigvuldiging. Sê telkens of die gegewe versameling 'n vektorruimte is. Indien nie, sê watter reël gebreek word en demonstreer dit. Indien wel, gee 'n geskikte basis vir die ruimte en sê wat sy dimensie is.

A vector space satisfies two rules: (1) closure under addition, and (2) closure under scalar multiplication. For each of the given sets, state whether it is a vector space or not. If not, say which rule was violated, and demonstrate it. If it is indeed a vector space, state the dimension of the space and give a suitable basis. $\boxed{\cancel{10}}$

$$A = \{\mathbf{a} \in \mathbb{R}^3 | a_1 > 0, a_2 + a_3 = 0\}$$
$$B = \{\mathbf{b} \in \mathbb{R}^3 | b_1 - 2b_2 + 4b_3 = 0\}$$
$$C = \{\mathbf{c} \in \mathbb{R}^3 | c_3(c_1 + c_2) = 0\}$$

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A is not a VS: Rule 2 is violated, e.g.

$$-1\begin{bmatrix}2\\5\\-5\end{bmatrix} = \begin{bmatrix}-2\\-5\\5\end{bmatrix} \quad -2 \not> 0$$

B is a VS: $b_1 = 2b_2 - 4b_3$

$$\underline{b} = \begin{bmatrix}2b_2 - 4b_3\\b_2\\b_3\end{bmatrix} = b_2\begin{bmatrix}2\\1\\0\end{bmatrix} + b_3\begin{bmatrix}-4\\0\\1\end{bmatrix}$$

basis $= \left\{\begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}-4\\0\\1\end{bmatrix}\right\}$ dimension $= 2$.

C is not a VS: Rule 1 is violated, e.g.

$$\underset{\text{in } C}{\begin{bmatrix}-1\\1\\5\end{bmatrix}} + \underset{\text{in } C}{\begin{bmatrix}2\\3\\0\end{bmatrix}} = \underset{\text{not in } C}{\begin{bmatrix}1\\4\\5\end{bmatrix}}$$

$$A = \begin{bmatrix} 2 & 1 & -3 & 5 \\ 6 & 3 & -9 & 19 \\ -2 & -1 & 3 & 11 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 12 \\ 44 \\ 20 \end{bmatrix}$$

18.

4 **$** Doen reghoekige LU-onbinding van die matriks A hierbo. Los dan op $A\mathbf{x} = \mathbf{b}$. Skryf die antwoorde in die spasies hieronder. Vind dan ook geskikte basisse vir die kolomruimte asook die nulruimte van A.

Do rectangular LU-decomposition of the matrix A above. Then solve $A\mathbf{x} = \mathbf{b}$. Write your answers in the spaces below. Also find suitable bases for the column space as well as the null space of A. $\boxed{15}$

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$$\begin{array}{cccc} 6 & 3 & -9 & 15 \end{array}$$
$$\begin{bmatrix} 2 & 1 & -3 & 5 \\ 6 & 3 & -9 & 19 \\ -2 & -1 & 3 & 11 \end{bmatrix}$$

$$\begin{matrix} 2 & 1 & -3 & 5 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 16 \end{matrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 2 & 1 & -3 & 5 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$L\underline{c} = \underline{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 44 \\ 20 \end{bmatrix}$$

$c_1 = 12$

$36 + c_2 = 44$, $c_2 = 8$

$-12 + 32 + c_3 = 20$

$c_3 = 0$

$$\underline{c} = \begin{bmatrix} 12 \\ 8 \\ 0 \end{bmatrix}$$

$U\underline{x} = \underline{c}$

$$\begin{bmatrix} \boxed{2} & 1 & -3 & 5 \\ 0 & 0 & 0 & \boxed{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ \lambda \\ \mu \\ x_4 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \\ 0 \end{bmatrix}$$

$4x_4 = 8$, $x_4 = 2$

$2x_1 + \lambda - 3\mu + 5(2) = 12$

$2x_1 = -\lambda + 3\mu + 12 - 10$

$= -\lambda + 3\mu + 2$

$x_1 = -\frac{1}{2}\lambda + \frac{3}{2}\mu + 1$

$$\underline{x} = \begin{bmatrix} -\frac{1}{2}\lambda + \frac{3}{2}\mu + 1 \\ \lambda \\ \mu \\ 2 \end{bmatrix}$$

$$\mathrm{L}=\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} \quad \mathrm{U}=\begin{bmatrix} 2 & 1 & -3 & 5 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{c}=\begin{bmatrix} 12 \\ 8 \\ 0 \end{bmatrix}$$

$$\mathbf{x}=\begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} \frac{3}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\text{basis}(\mathcal{C}(\mathcal{A}))=\begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix}\begin{bmatrix} 5 \\ 19 \\ 11 \end{bmatrix}, \quad \text{basis}(\mathcal{N}(\mathcal{A}))=\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} \frac{3}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Vir die basisse mag daar meer spasies gegee word as nodig.

For the bases there may be more spaces given than necessary.

5 Herlei die vorm van die projeksie-matriks op die kolom-ruimte van A, naamlik $P = A(A^TA)^{-1}A^T$, waar A vol rang het. Jy mag vir die doel van die herleiding aanvaar dat A slegs twee kolomme in $\mathbb{R}^3$ het, en jou herleiding moet van 'n skets vergesel wees.

Derive the form of the projection matrix on the column space of A, i.e. $P = A(A^TA)^{-1}A^T$, where A has full rank. For the derivation, you may assume that A has only two columns in $\mathbb{R}^3$ and a sketch must accompany your derivation. $\boxed{10}$

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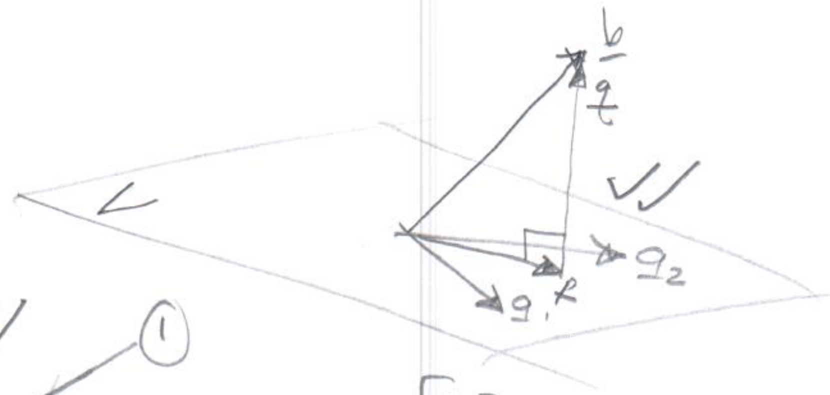
Let $A = \begin{bmatrix} | & | \\ \underline{a}_1 & \underline{a}_2 \\ | & | \end{bmatrix}$

and $\{\underline{a}_1, \underline{a}_2\}$ spans the plane V. Let



$\underline{p} = \lambda \underline{a}_1 + \mu \underline{a}_2$

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or $\underline{p} = \begin{bmatrix} | & | \\ \underline{a}_1 & \underline{a}_2 \\ | & | \end{bmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = A\underline{x}$, where $\underline{x} = \begin{bmatrix} \lambda \\ \mu \end{bmatrix}$

be the projection of $\underline{b}$ onto V.

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Let $\underline{q} = \underline{b} - \underline{p} = \underline{b} - A\underline{x}$. For a projection $\underline{q}$ must be orthogonal to both $\underline{a}_1$ and $\underline{a}_2$ or

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$\left.\begin{matrix} \underline{a}_1^T\underline{q} = 0 \\ \underline{a}_2^T\underline{q} = 0 \end{matrix}\right\}$ or $A^T\underline{q} = \underline{0}$ therefore

$A^T(\underline{b} - A\underline{x}) = \underline{0}$ or $A^TA\underline{x} = A^T\underline{b}$

Premultiply by $(A^TA)^{-1}$: $\underline{x} = (A^TA)^{-1}A^T\underline{b}$

then $\underline{p} = A\underline{x} = A(A^TA)^{-1}A^T\underline{b}$

or $\underline{p} = P\underline{b}$ where $P = A(A^TA)^{-1}A^T\underline{b}$

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6(a) As $P = A(A^TA)^{-1}A^T$, bewys dat $P^2 = P$.

If $P = A(A^TA)^{-1}A^T$, prove that $P^2 = P$. $\boxed{2}$

6(b) Bewys dat $P_c = I - P$ ook 'n projeksiematriks is, d.w.s. $(P_c)^2 = P_c$.

Prove that $P_c = I - P$ is also a projection matrix, i.e. $(P_c)^2 = P_c$. $\boxed{2}$

6(c) Bewys dat $P\mathbf{b}$ en $P_c\mathbf{b}$ ortogonaal is vir enige $\mathbf{b} \in \mathbb{R}^n$.

Prove that $P\mathbf{b}$ and $P_c\mathbf{b}$ are orthogonal for any $\mathbf{b} \in \mathbb{R}^n$. $\boxed{2}$

6(d) Bewys dat 'n refleksiematriks $H = 2P - I$, die eienskap het dat $H^3 = H$.

Prove that a reflection matrix $H = 2P - I$ has the property that $H^3 = H$. $\boxed{2}$

6(e) Bewys dat beide P en H simmetries is.

Prove that both P and H are symmetric. $\boxed{2}$

6(f) Bewys dat H lengtes behou, d.w.s. $\|H\mathbf{b}\|^2 = \|\mathbf{b}\|^2$.

Prove that H conserves lengths, i.e. $\|H\mathbf{b}\|^2 = \|\mathbf{b}\|^2$. $\boxed{2}$

6(g) Bewys dat as die hoek tussen $\mathbf{a}$ en $\mathbf{b}$ θ is, dan sal die hoek tussen $H\mathbf{a}$ en $H\mathbf{b}$ ook θ wees.

Prove that if the angle between $\mathbf{a}$ and $\mathbf{b}$ is θ, then the angle between $H\mathbf{a}$ and $H\mathbf{b}$ is also θ. $\boxed{6}$

(a) $P^2 = (A(A^TA)^{-1}A^T)(A(A^TA)^{-1}A^T)$
$= A\underbrace{(A^TA)^{-1}A^TA}_{I}(A^TA)^{-1}A^T = A(A^TA)^{-1}A^T = P$

(b) $(I-P)^2 = I - 2P + P^2 = I - 2P + P = I - P$

(c) $(P\underline{b})^T(P_c b) = \underline{b}^T P(I-P)\underline{b} = \underline{b}^T(P-P^2)\underline{b}$
$= \underline{b}^T(P-P)\underline{b} = \underline{b}^T O \underline{b} = 0$

(d) $H^2 = (2P-I)(2P-I) = 4P^2 - 4P + I$
$= 4P - 4P + I = I$

$H^3 = H^2H = IH = H$

(e) $P^T = (A(A^TA)^{-1}A^T)^T = (A^T)^T((A^TA)^{-1})^TA^T$
$= A((A^TA)^T)^{-1}A^T = A(A^TA)^{-1}A^T = P$

$H^T = (2P-I)^T = 2P^T - I^T = 2P - I = H.$

Einde van toets / *End of test*

(f) $\|H\underline{b}\|^2 = (H\underline{b})^T(H\underline{b})$

$= \underline{b}^T H^T H \underline{b}$

$= \underline{b}^T H^2 \underline{b} = \underline{b}^T I \underline{b} = \underline{b}^T\underline{b} = \|\underline{b}\|^2$

(g) $\underline{a}^T\underline{b} = ab\cos\theta$ ← angle between $\underline{a}$ and $\underline{b}$

length of $\underline{a}$ length of $\underline{b}$

$(H\underline{a})^T(H\underline{b}) = \underline{a}^T H^T H \underline{b} = \underline{a}^T H^2 \underline{b}$

$= \underline{a}^T I \underline{b} = \underline{a}^T\underline{b}$

$= a\,b\cos\theta$

length of $\underline{b}$, but also length of $H\underline{b}$ (from (f))

length of $\underline{a}$, and also length of $H\underline{a}$ (from (f))