TW 214 — TOETS 2 - VOORBEREIDING — 2017

TEST 2 - PREPARATION

Die toets gaan oor die volgende onderwerpe:

- REFLEKSIES (GEKROMDE SPIEËLS)
- QR-ONTBINDING
- DETERMINANTE
- EIEWAARDES EN EIEWAARDE-ONTBINDING ($A = S\Lambda S^{-1}$)
- STELSELS VAN VERSKILVERGELYKINGS
- STELSELS VAN DIFFERENSIAALVERGELYKINGS
- EIEWAARDES VAN SIMMETRIESE MATRIKSE ($A = Q\Lambda Q^T$)
- KWADRATIESE KROMMES
- DIE SVD
- ROTASIES IN 3D

The test will be on the following topics

- *REFLECTIONS (CURVED MIRRORS)*
- *QR-DECOMPOSITION*
- *DETERMINANTS*
- *EIGENVALUES AND EIGENVALUE DECOMPOSITION ($A = S\Lambda S^{-1}$)*
- *SYSTEMS OF DIFFERENCE EQUATIONS*
- *SYSTEMS OF DIFFERENTIAL EQUATIONS*
- *EIGENVALUES OF SYMMETRIC MATRICES ($A = Q\Lambda Q^T$)*
- *QUADRATIC CURVES*
- *THE SVD*
- *ROTATIONS in 3D*

Vir hierdie toets moet jy die volgende kan doen:

For this test you must be able to do the following:

REFLEKSIES (GEKROMDE SPIEËLS)

— Die eienskappe van die refleksiematrikse van die vorm $2P - I$ kan aantoon.

— Die verwante refleksiematriks vir 'n spieël kan vind (uit die projeksiematriks op die spieël, 2D of 3D), en sy eienskappe kan aantoon.

— Probleme met invallende ligstrale op gekromde spieëls (slegs in 2D) kan doen waar refleksiematrikse gebruik word.

REFLECTIONS (CURVED MIRRORS)

— *Be able to prove the properties of a reflection matrix of the form $2P - I$.*

— *Find the corresponding reflection matrix for a mirror (from either the projection matrix on the mirror, 2D or 3D) and prove its properties.*

— *Do problems with light beams on curved mirrors (2D only) where reflection matrices are used.*

KLEINSTE-KWADRATE OPLOSSING, ORTOGONALE MATRIKSE EN QR-ONTBINDING

— Kan aantoon hoe die kleinste-kwadrate oplossing gevind word as $x = (A^TA)^{-1}A^T\mathbf{b}$, met verduidelikings van wat bedoel word met "kleinste-kwadrate oplossing".

— Die kleinste-kwadrate-oplossing van $A\mathbf{x} \approx \mathbf{b}$ kan vind as $\mathbf{b}$ nie in A se kolomruimte lê nie.

— Beste lyn, of polinoom en/of ander funksie (wat lineêr afhang van die koëffisiënte) deur 'n gegewe stel punte kan vind met behulp van kleinste kwadrate oplossing van oorbepaalde stelsel.

— Moet die algemene formules vir m en c van die "beste" lyn deur 'n aantal punte kan herlei, of soortgelyke formule vir ander krommes (slegs gevalle waar A^TA 'n 2×2-matriks is, kan gevra word).

— Eienskappe van ortogonale matrikse ken en kan aantoon: $Q^TQ = ..,\quad QQ^T = ...,$ behoud van lengtes en hoeke. Ook die eienskappe van 'n reghoekige matriks met ortogonale kolomme, $\bar{Q}$ moet aangetoon kan word.

— QR-ontbinding van 'n vierkantige matriks kan doen en 'n vierkantige stelsel daarmee kan oplos.

— Reghoekige $\overline{Q}$R-ontbinding van 'n portretvormige matriks A kan doen en die kleinste-kwadrate-oplossing van die oorbepaalde stelsel geassosieer met A daarmee kan vind.

— Moet ook die projeksiematriks kan vind wat op 'n portretvormige A se kolomruimte projekteer deur $\overline{Q}$R-ontbinding te gebruik.

LEAST SQUARES SOLUTION, ORTHOGONAL MATRICES AND QR-DECOMPOSITION

— *Show how the least squares solution is found as* $\mathbf{x} = (A^TA)^{-1}A^T\mathbf{b}$*, with explainations of what is meant by "least squares solution".*

— *Find the least squares solution of* $A\mathbf{x} \approx \mathbf{b}$ *if* $\mathbf{b}$ *does not lie in the column space of* A*.*

— *Find the best line, or polynomial and/or other function (that depends linearly on the coefficients) through a given set of points with the help of least squares solution of an overdetermined system.*

— *Must be able to derive the general formulas for* m *and* c *of the "best" line through a number of points, or similar formulas for other curves (only the cases where* A^TA *is a* 2×2 *matrix, will be asked).*

— *Must know and be able to derive the properties of orthogonal matrices, e.g.* $Q^TQ = ..,\quad QQ^T = ...,$ *conservation of lengths and angles. This also applies to the properties of a rectangular matrix with orthonormal columns,* $\bar{Q}$*.*

— *Must be able to do QR-decomposition of a square matrix and be able to find the solution of a square system by using the QR decomposition.*

— *Must be able to do rectangular* $\overline{Q}$*R-decomposition of a portrait shaped matrix* A *and be able to find the least squares solution of the overdetermined system associated with* A *using this* $\overline{Q}$*R decomposition.*

— *Must also be able to find the projection matrix that projects on the column space of a portrait shaped* A *by using the* $\overline{Q}$*R-decomposition of* A*.*

DIE DETERMINANT

Jy sal nie direk getoets word vir kennis van determinante nie, maar alles wat jy in Wiskunde geleer het (bv. hoe om det van 'n 3×3 matriks te bereken, en eienskappe soos (a) *as daar 'n nul-ry is, is det=0*, (b) *as rye omgeruil word verander det se teken, ens. ...* mag benodig word (dus moet dit geken word).

THE DETERMINANT

You will not be tested directly for knowledge of determinants, but, all that you learned in Maths (e.g. how to calculate det of a 3×3 matrix, as well as properties such as e.g. (a) if there is a zero row, det=0, (b) when two rows are exchanged, det changes sign, etc. ... may be needed (so it must be known).

EIEWAARDES EN EIEVEKTORE

— Eiewaardes en eievektore van gegewe 2×2 en 3×3 matrikse kan vind, en veral weet hoe om dit makliker te vind as een eievektor of een eiewaarde gegee is. Ook die gebruik van $\det(A)$ en $\mathrm{tr}(A)$, kan die vind van eiewaardes vergemaklik.

— Diagonaalvorm van 'n matriks kan vind, d.w.s. $A = S\Lambda S^{-1}$

— As A simmetries is, die spesiale diagonaalvorm met ortogonale matrikse kan vind, $A = Q\Lambda Q^T$, ook as A gelyke eiewaardes het.

— Die verband tussen die som en produk van die eiewaardes met die determinant en spoor van die matriks kan aantoon vir die 2×2 geval.

— Spesiale eienskappe van die eiewaardes en eievektore van *simmetriese* matrikse ken en kan aantoon, bv. eiewaardes is reëel en $(\lambda_k - \lambda_j)\mathbf{x}_k^T\mathbf{x}_j = 0$ en dit kan interpreteer.

— Die verband wat die eiewaardes en eievektore van A, A^n, A^{-1} en $A - \alpha I$ met mekaar het ken en kan aantoon.

EIGENVALUES AND EIGENVECTORS

— *Be able to find eigenvalues and eigenvectors of a given 2×2 and 3×3 matrix, especially how to find it easier if one eigenvector or one eigenvalue is given. The use of $\det(A)$ and $\mathrm{tr}(A)$ can also simplify the finding of eigenvalues.*

— *Be able to find the diagonal form of a matrix, i.e. $A = S\Lambda S^{-1}$*

— *If A is symmetric, be able to find the special diagonal form with orthogonal matrices, $A = Q\Lambda Q^T$, also if A has equal eigenvalues.*

— *Show the relationship between the sum and product of the eigenvalues, and the determinant and trace of the matrix, for the 2×2 case.*

— *Know and derive special properties of the eigenvectors and eigenvalues of* symmetric *matrices, e.g. eigenvalues are real and $(\lambda_k - \lambda_j)\mathbf{x}_k^T\mathbf{x}_j = 0$ and be able to interpret it.*

— *Know and derive the relationship that the eigenvalues and eigenvectors of A, A^n, A^{-1} and $A - \alpha I$ have with each other.*

STELSELS VAN VERSKILVERGELYKINGS

— Herlei die oplossing van 'n stelsel van verskilvergelykings van die vorm $\mathbf{u}_{n+1} = A\mathbf{u}_n$. (Dit is $\mathbf{u}_k = S\Lambda^k S^{-1}\mathbf{u}_0$.)

— Stelsels van eerste orde verskilvergelykings kan oplos (slegs 2×2-stelsels).

— Tweede orde verskilvergelykings kan skryf as stelsels van eerste orde verskilvergelykings en kan oplos.

— Vir praktiese probleme wat lei tot 'n stelsel van verskilvergelykings, die stelsel kan neerskryf en uitdruk in matriksvorm. (Diskrete groei-modelle, getalle-reekse, ens.)

— Limiete kan vind as $k \to \infty$, waar van toepassing.

SYSTEMS OF DIFFERENCE EQUATIONS

— *Derive the solution of a system of difference equations of the form* $\mathbf{u}_{n+1} = A\mathbf{u}_n$. *(It is* $\mathbf{u}_k = S\Lambda^k S^{-1}\mathbf{u}_0$.*)*

— *Solve systems of first order difference equations (*2×2*-systems only).*

— *Be able to write second order difference equations as systems of first order difference equations and solve it.*

— *For practical problems that are expressible as a system of difference equation, you must be able to wrtie down the system and express it in matrix form (Discrete growth models, number sequences, etc.)*

— *Be able to find limits as* $k \to \infty$ *where applicable.*

STELSELS VAN DIFFERENSIAALVERGELYKINGS

— Herlei die oplossing van 'n stelsel van differensiaalvergelykings van die vorm $\dot{\mathbf{u}} = A\mathbf{u}$. (Dit is $\mathbf{u}(t) = Se^{\Lambda t}S^{-1}\mathbf{u}(0)$.)

— Stelsels van eerste orde differensiaalvergelykings kan oplos (slegs 2×2-stelsels).

— Tweede orde differensiaalvergelykings kan skryf as stelsels van eerste orde differensiaalvergelykings en kan oplos.

— In praktiese probleme wat 'n toepassing van differensiaalvergelykings is, die stelsel kan neerskryf en ook uitdruk in matriksvorm. (Tenks met oplossings in., Meganiese stelsels, ens. Genoeg wenke sal gegee word.)

— Limiete kan vind as $t \to \infty$, maksima of minima van oplossings kan vind, of die tyd wanneer 'n maksimum of minimum voorkom kan vind.

SYSTEMS OF DIFFERTIAL EQUATIONS

— *Derive the solution of a system of differential equations of the form* $\dot{\mathbf{u}} = A\mathbf{u}$. *(It is* $\mathbf{u}(t) = Se^{\Lambda t}S^{-1}\mathbf{u}(0)$.*)*

— *Solve systems of first order differential equations (*2×2*-systems only).*

— *Be able to write second order differential equations as systems of first order differential equations and solve it.*

— *In practical problems that are applications of differential equations, you must be able to write down the system and also express it in matrix form. (Tanks with solutions, mechanical systems, etc. Enough hints will be given.)*

— *Be able to find limits as* $t \to \infty$*, or find maxima or minima or find the time when a maximum or minimum occurs.*

KWADRATIESE KROMMES

— Moet die formules $\mathbf{r} = -\frac{1}{2}A^{-1}\mathbf{b}$, $d = \mathbf{r}^T A\mathbf{r} - c$ om 'n algemene kwadratiese kromme van die vorm $\mathbf{u}^T A\mathbf{u} + \mathbf{b}^T\mathbf{u} + c = 0$, om te skakel in die vorm $\mathbf{v}'^T A\mathbf{v}' = d$, kan aflei, en die fisiese betekenisse van $\mathbf{u}$, $\mathbf{v}'$, $\mathbf{r}$ kan verduidelik.

— Moet die koördinate van die middelpunt, die rotasiehoek en die lengtes van die semihoofasse van die figuur voorgestel deur 'n kwadratiese vergelyking, kan vind en 'n rowwe skets van die figuur kan maak met al die inligting daarop aangedui. Slegs ellipse of hiperbole kan gevra word.

QUADRATIC CURVES

— *Derive the formulas* $\mathbf{r} = -\frac{1}{2}A^{-1}\mathbf{b}$, $d = \mathbf{r}^T A\mathbf{r} - c$ *to transform a general quadratic curve of the form* $\mathbf{u}^T A\mathbf{u} + \mathbf{b}^T\mathbf{u} + c = 0$, *to the form* $\mathbf{v}'^T A\mathbf{v}' = d$, *and explain the physical meaning of* $\mathbf{u}$, $\mathbf{v}'$, $\mathbf{r}$.

— *Find the coordinates of the centre, rotation angle and the lengths of the semi-axes of the figure represented by a quadratic equation, and draw a rough sketch of the figure, containing this information. Only ellipses or hyperbolas will be asked.*

DIE SVD

— Moet $2 \times n$ en $n \times 2$ matrikse kan SVD-ontbind, met $n = 2, 3, 4$ of 5. Beide volle SVD en gereduseerde SVD kan gevra word.

— Die interpretasie ken van wat elk van die faktore U, Σ en V^T doen as 'n matriks met 'n vektor vermenigvuldig word (roteer-skaal-roteer).

— Moet ortogonale basisse van die vier fundementele ruimtes (kolom-, ry-, nul-, en linksnul-ruimte) van 'n matriks kan neerskryf as die SVD van 'n matriks beskikbaar is. Moet ook kan aantoon waar hierdie verband vandaan kom.

— Moet die verband tussen die SVD en rang-1 ontbinding van 'n matriks kan beskryf.

— Jy moet die kleinste-kwadrate oplossing van 'n reghoekige stelsel kan vind met behulp van die SVD.

THE SVD

— *SVD-decompose* $2 \times n$ *and* $n \times 2$ *matrices, with* $n = 2, 3, 4$ *or* 5.

— *Know the interpretation of what each of the factors* U, Σ *and* V^T *does, when a matrix is multiplied with a vector (rotate-scale-rotate).*

— *Write down the orthogonal basis of each of the four fundamental spaces (column, row, null, and left-nullspace) of a matrix, if the SVD of a matrix is available. Also be able to show where this relationship comes from.*

— *Must be able to describe the relationship between the SVD and the rank-1 decomposition of a matrix.*

— *You must be able to find the least squares solution of a rectangular system by using the SVD.*

DIE 3×3-ROTASIE-MATRIKS

— Moet die verband tussen die kruisproduk-matriks A en die rotasie-as $\mathbf{a}$ ken en kan neerskryf.

— Moet die eienskappe van A en A^2 kan aantoon (simmetrie, spoor, en $\mathbf{a}\mathbf{a}^T = I + A^2$).

— Moet die volgende verband tussen die rotasiematriks Q en die rotasie-as $\mathbf{a}$ en hoek θ ken en kan herlei uit eerste beginsels: $Q = I + A\sin\theta + A^2(1-\cos\theta)$.

— Moet die vorm $Q = I + A\sin\theta + A^2(1-\cos\theta)$ kan gebruik om die eienskappe van Q aan te toon.

— Moet die eienskappe van die eiewaardes en eievektore van Q en A kan aantoon en kan interpreteer.

— Moet die formules $\cos\theta = \frac{1}{2}(\mathrm{Tr}(Q) - 1)$ en $A = (Q - Q^T)/(2\sin\theta)$ kan herlei, en kan gebruik om die rotasie-as en rotasie-hoek van 'n rotasie-matriks te kan vind. Die rotasie-matriks Q moet ook gevind kan word as θ en $\mathbf{a}$ gegee is.

— Eenvoudige toepassings kan gevra word.

Spesiale insig-vrae

Vrae of gedeeltes van vrae wat nie presies in een van die katergorieë hierbo val nie, maar wat hoofsaaklik die kennis oor lineêre algebra toets wat jy in hierdie kursus geleer het, *saam met insig*, kan gevra word. Hierdie vrae sal egter nie meer as 15% van die vraestel uitmaak nie.

THE 3×3 ROTATION MATRIX

— *Know and be able to write down the relationship between the cross-product matrix A and the rotation axis $\mathbf{a}$.*

— *Prove the properties of A and A^2 (symmetry, trace, and $\mathbf{a}\mathbf{a}^T = I + A^2$).*

— *Know and derive the following relationship between the rotation matrix Q and the rotation axis $\mathbf{a}$ and the angle θ, from first principles: $Q = I + A\sin\theta + A^2(1-\cos\theta)$.*

— *Must be able to use the form $Q = I + A\sin\theta + A^2(1-\cos\theta)$ to prove properties of Q.*

— *Be able to derive and interpret the properties of the eigenvalues and eigenvectors of Q and A.*

— *Derive the formulae $\cos\theta = \frac{1}{2}(\mathrm{Tr}(Q) - 1)$ and $A = (Q - Q^T)/(2\sin\theta)$, and use them to find the rotation axis and rotation angle of a rotation matrix. Also, be able to obtain the rotation matrix if θ and $\mathbf{a}$ are given.*

— *Simple applications may be asked.*

Special insight questions

Questions or parts of questions that do not clearly fall into any of the above categories, but that mainly tests the knowledge of linear algebra that you acquired during this module, with insight, *may be asked. However, these questions will not contribute to more than 15% of the paper.*