

Chapter 5: Stochastic differential equations

• Jacobs, Chaps 2, 3

• GS, Chap. 13

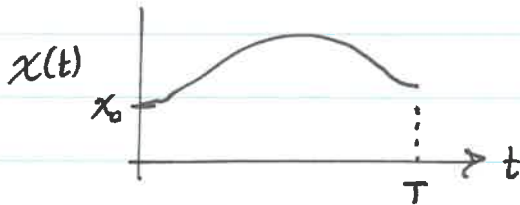
5.1 Introduction

Ordinary differential equations
ODEs

$$\begin{cases} \frac{dx(t)}{dt} = f(x(t), t) \\ x(0) = x_0 \end{cases}$$

$$\dot{x} = f(x, t)$$

"force"
deterministic evolution



Stochastic differential equations
SDEs

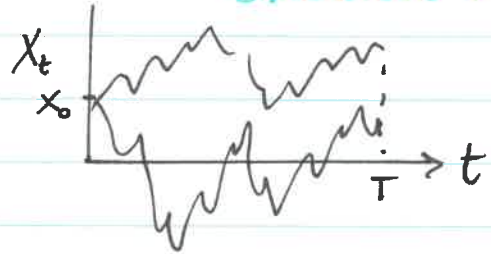
$$\begin{cases} \frac{dX_t}{dt} = f(X_t, t) + \underbrace{\xi_t}_{\text{noise}} \\ X_0 = x_0 \end{cases}$$

random variable

$$\dot{X} = f(X, t) + \xi$$

"force" noise
deterministic

Stochastic evolution



5.2 Ordinary differential equations (ODEs)

• 1st order ODE:

$$\dot{\vec{x}}(t) = \vec{f}(\vec{x}(t)) \quad , \quad \vec{x}(0) = \vec{x}_0$$

• State: $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \quad \vec{x}(0) = \begin{pmatrix} (x_0)_1 \\ (x_0)_2 \\ \vdots \\ (x_0)_n \end{pmatrix}$

• Force or drift: $\vec{f} = \begin{pmatrix} f_1(\vec{x}) \\ f_2(\vec{x}) \\ \vdots \end{pmatrix} = \begin{pmatrix} f_1(x_1, \dots, x_n) \\ f_2(x_1, \dots, x_n) \\ \vdots \end{pmatrix}$

• Any n^{th} order ODE for $x(t) \in \mathbb{R}$ can be transformed to 1st order ODE for $\vec{y}(t) \in \mathbb{R}^n$.

Example: Newton's equation

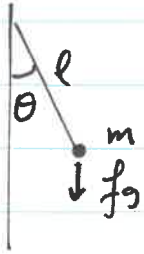
$$F = ma = m\dot{v} = m\ddot{x}$$

$$\dot{x}(t) = v(t) = f_x(x, v)$$

1st order 2nd order

$$\dot{v}(t) = F/m = f_v(x, v)$$

Example: Pendulum



$$m\ddot{\theta} + \frac{mg}{l} \sin \theta = 0$$

$$\ddot{\theta} + \omega^2 \sin \theta = 0$$

$\omega = \sqrt{\frac{g}{l}}$ natural frequency

$$\Rightarrow \begin{aligned} \dot{\theta} &= \varphi & &= f_{\theta}(\theta, \varphi) \\ \dot{\varphi} &= \ddot{\theta} = -\omega^2 \sin \theta & &= f_{\varphi}(\theta, \varphi) \end{aligned}$$

Angular velocity

$$\Rightarrow \begin{pmatrix} \dot{\theta} \\ \dot{\varphi} \end{pmatrix} = \begin{pmatrix} \varphi \\ -\omega^2 \sin \theta \end{pmatrix} \quad \begin{aligned} f_{\theta}(\theta, \varphi) &= \varphi \\ f_{\varphi}(\theta, \varphi) &= -\omega^2 \sin \theta \end{aligned}$$

5.3 Numerical solutions of ODEs

• Matlab: ode45 → Runge-Kutta

① Define f as function

Function $dxdt = f(t, x)$

$dxdt = \text{zeros}(2, 1);$

$dxdt(1) = \dots;$

$dxdt(2) = \dots;$

end

or

Function $dxdt = f(t, x)$

$dxdt = [\dots \dots];$

end

② Use f in solver

$tspan = [0, 10];$

$x10 = \dots;$

$x20 = \dots;$

$[T, X] = \text{ode45}(@f, tspan,$

$[x10 \ x20]);$

$\text{plot}(T, X(:, 1), T, X(:, 2))$

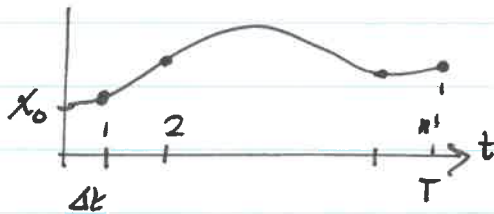
time vector x_1 vector, traj in time

Python: odeint from scipy.integrate

```
def f(x, t):
    : parameters
    dx1dt = ...
    dx2dt = ...
    return [dx1dt, dx2dt]
```

```
t_final = 10.0
dt = 0.01
t_span = np.linspace(
    0, t_final, int(t_final/dt))
x0 = [..., ...]
x = odeint(f, x0, t_span)
plt.plot(t_span, x[:, 0])
```

Euler scheme:



$$\begin{cases} \frac{dx(t)}{dt} = f(x(t)) \\ x(0) = x_0 \end{cases}$$

- Approximate $\{x(t)\}_{t=0}^T$ with n points $\{x_i\}_{i=0}^n$
- $n = T/dt$

$$\Delta x(t) = x(t + dt) - x(t) = f(x(t)) dt$$

$$\Rightarrow x(t + dt) = x(t) + f(x(t)) dt + O(dt^2)$$

Differentials

$$\Rightarrow x_{i+1} = x_i + f(x_i) dt$$

next present

Euler scheme

Code:

```
T = 10.0
dt = 0.01
n = int(T/dt)
x = zeros(n+1)
x[0] = ...
for i = 0:n-1
    x[i+1] = x[i] + f(x[i]) * dt
end
tspan = [0:dt:T]
plot(tspan, x)
```

Note: Total error

$$\text{Err} = \text{error per step} \times \text{no steps}$$

$$= dt^2 \times \frac{T}{dt} \sim dt$$

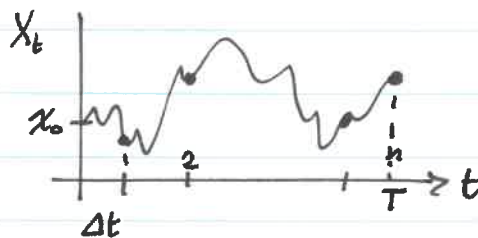
Note: Other methods:

- Mid point
- Implicit
- Verlet
- Runge-Kutta

5.4 Stochastic differential equations (SDEs)

$$\begin{cases} \frac{dX_t}{dt} = f(X_t) + \xi_t \\ X_0 = x_0 \end{cases}$$

def. noise



Discretization:

$$\Delta X_t = f(X_t) \Delta t + \xi_t \Delta t$$

$$= \underbrace{f(X_t) \Delta t}_{\text{Euler}} + \underbrace{\Delta W_t}_{\text{Increment of Wiener process/BM}}$$

↳ Gaussian white noise

$$\Rightarrow X_{t+\Delta t} = X_t + f(X_t) \Delta t + \Delta W_t$$

$$\Delta W_t \sim \mathcal{N}(0, \Delta t) = \sqrt{\Delta t} \mathcal{N}(0, 1)$$

$$\Rightarrow \begin{cases} X_{i+1} = X_i + f(X_i) \Delta t + \sqrt{\Delta t} Z_i \\ Z_i \sim \mathcal{N}(0, 1) \end{cases}$$

Euler-Maruyama scheme

Note: Noise chosen to have variance \sim time. Other choices of noise possible: e.g., colored noise (correlated noises), heavy noise, Poisson noise, etc.

Ito notation for SDEs:

$$\begin{aligned} \dot{X}_t &= f(X_t) + \xi_t \\ &= f(X_t) + \frac{dW_t}{dt} \end{aligned}$$

But W_t is nowhere differentiable! dW_t/dt doesn't make sense. Write instead as

$$dX_t = f(X_t) dt + dW_t$$

Same as $\Delta X_t = f(X_t) \Delta t + \Delta W_t$

Difference equation well defined.

• Solution of SDE:

$$dX_t = f(X_t)dt + dW_t \quad \Rightarrow \quad X_t = X_0 + \int_0^t f(X_s)ds + \int_0^t dW_s$$

$$= X_0 + \int_0^t f(X_s)ds + W_t$$

implicit eq.

• Example: Add Gaussian white noise to linear eq.

$$\begin{cases} \dot{x}(t) = -\gamma x(t) \\ x(0) = x_0 \end{cases}$$

$$\hookrightarrow \dot{x}(t) = -\gamma x(t) + \xi_t$$

Noisy ODE

$$\text{i.e. } dX_t = -\gamma X_t dt + dW_t$$

SDE $dW_t \sim \mathcal{N}(0, dt)$

$$t_{\text{final}} = 10.0$$

$$dt = 0.01$$

$$n = \text{int}(t_{\text{final}}/dt)$$

$$x = \text{zeros}(n+1)$$

$$x[0] = \dots$$

for i in range(n):

$$x[i+1] = x[i] + \text{gamma} * x[i] * dt + \sqrt{dt} * \text{randn}()$$

end

$$t_{\text{span}} = [0 : dt : t_{\text{final}}]$$

$$\text{plot}(t_{\text{span}}, x)$$

• Exercise: Change dt , x_0 , t_{final} , ...

• Example: Where to put noise in a Newton equation?

$$m a = m \ddot{x} = \underbrace{\text{Force}}_{F_{\text{tot}}} + \text{noise}$$

$$\Rightarrow \dot{x} = v \quad \leftarrow \text{no noise on velocity}$$

$$\dot{v} = \frac{F_{\text{tot}}}{m} = \frac{1}{m} (\text{Force} + \text{noise}) \quad \leftarrow \text{noise in force/acceleration}$$

Example: Noisy pendulum

$$d\theta_t = \varphi_t dt$$

$$d\varphi_t = -\omega^2 \sin\theta_t dt + dW_t$$

← no noise

← Noise in acceleration

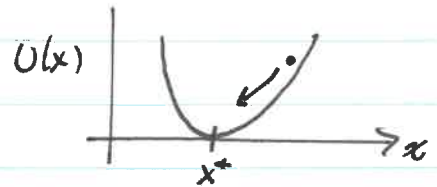
Example: Gradient dynamics

$$\cdot \vec{x}(t) \in \mathbb{R}^n$$

$$\cdot \dot{\vec{x}}(t) = -\nabla U(\vec{x}(t))$$

$$\cdot \text{Fixed point: } \dot{\vec{x}}^* = 0 \iff \vec{x}^* \text{ is a min of } U(\vec{x})$$

(assuming not starting at critical point)
↑
other



· SDE version:

$$dX_t = -\nabla U(X_t) dt + \sigma dW_t$$

← noise amplitude

See CWS.

5.5 Fokker-Planck equation

· SDE:

$$\begin{cases} dX_t = F(X_t) dt + \sigma(X_t) dW_t \\ X_0 = x_0 \end{cases}$$

$$X_t \in \mathbb{R}^n$$

$$F: \mathbb{R}^n \rightarrow \mathbb{R}^n$$

$$W_t \in \mathbb{R}^m$$

$$\sigma \text{ } n \times m$$

· Most cases: $m=n$ so σ is square matrix

· Consider here: σ is constant matrix

· Probability density of X_t : $p(x, t) = p(X_t = x)$

· X_t continuous space + continuous time

· X_t called a diffusion

· What is the evolution equation for $p(x, t)$?

• Evolution equation:

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} p(x,t) = -\nabla \cdot (F(x) p(x,t)) + \frac{1}{2} \nabla \cdot D \nabla p(x,t) \\ p(x,0) = \delta(x_0) \end{array} \right.$$

$$n \left\{ \begin{array}{l} \frac{\partial}{\partial t} p(x,t) = -\sum_i \frac{\partial}{\partial x_i} F_i(x) p(x,t) + \frac{1}{2} \sum_{ij} \frac{\partial}{\partial x_i} D_{ij} \frac{\partial}{\partial x_j} p(x,t) \\ p(x,0) = \delta(x_0) \end{array} \right.$$

• $D = \sigma \sigma^T$

• Called: Fokker-Planck or forward Kolmogorov equation.

• Similar to master equation: $\partial_t p = L p$

↳ linear operator

• Diagonal D: $D = \sigma^2 \mathbb{1}$

$$\Rightarrow \partial_t p = -\nabla \cdot F p + \frac{\sigma^2}{2} \Delta p$$

• 1D case: $\partial_t p = -\frac{\partial}{\partial x} F(x) p + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} p$

• Proof/derivation: Jacobs Sec 7.1 p. 103, Pavliotis Sec 2.5 p. 18.

• Example: BM $F=0$, $\sigma=1$

$$\partial_t p = \frac{1}{2} \Delta p$$

Heat equation

$$p_{BM}(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$$

• Example: Drifted BM $F=\mu$ $\sigma>0$

$$\partial_t p = -\partial_x \mu p + \frac{\sigma^2}{2} \partial_{xx} p$$

$$= -\mu \partial_x p + \frac{\sigma^2}{2} \partial_{xx} p$$

$$p(x,t) = p_{BM}(x-\mu t, t)$$

• Example: Langevin equation

• Noisy ODE: $m \frac{d^2 x(t)}{dt^2} = F(x) - \gamma \frac{dx(t)}{dt} + \sigma \frac{d\xi(t)}{dt}$

force
friction
Gaussian white noise

-γv

• SDE :

$$dX_t = V_t dt$$

$$dV_t = \frac{F}{m} dt - \frac{\gamma}{m} V_t dt + \sigma_v dW_t$$

• State: (X_t, V_t)

• Distribution: $p(x, v, t) = p(X_t = x, V_t = v)$

• Fokker-Planck eq.:

$$\partial_t p(x, v, t) = -\partial_x F_x p(x, v, t) - \partial_v F_v p(x, v, t) + \frac{\sigma_x^2}{2} \partial_{xx} p(x, v, t) + \frac{\sigma_v^2}{2} \partial_{vv} p(x, v, t)$$

$$\Rightarrow \partial_t p = \underbrace{-\partial_x v p - \partial_v \left(\frac{F}{m} - \frac{\gamma}{m} v \right) p}_{\text{advection terms}} + \underbrace{\frac{\sigma_v^2}{2} \partial_{vv} p}_{\text{diffusion only on } v}$$

• Example: Overdamped equation. Neglect \ddot{x} term.

$$\Rightarrow 0 = F - \gamma \dot{x} + \sigma \xi_t$$

$$\Rightarrow dX_t = \frac{F}{\gamma} dt + \frac{\sigma}{\gamma} dW_t$$

• Force: $F = F(x)$ (other examples...)
 $= -\nabla U$ (gradient)

• Distribution: $p(x, t) = p(X_t = x)$ x only
no v-

• F-P eq.:

$$\partial_t p = -\partial_x \frac{F}{\gamma} p + \frac{\sigma^2}{2\gamma^2} \partial_{xx} p$$

5.6 Ergodic diffusions

- Stationary distribution: $P_S(x)$ such that $\partial_t P_S(x) = 0$

$$\Rightarrow -\nabla \cdot F P_S + \frac{1}{2} \nabla \cdot D \nabla P_S = 0$$

Diagonal case: $-\nabla \cdot F P_S + \frac{\sigma^2}{2} \Delta P_S = 0$

1D case: $-\partial_x F P_S + \frac{\sigma^2}{2} \partial_{xx} P_S = 0$

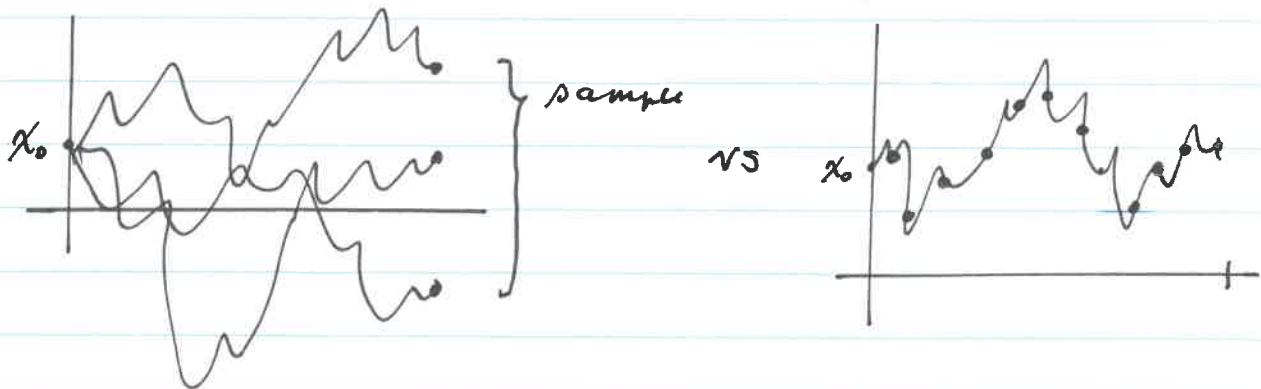
- Ergodic process:

$$\lim_{t \rightarrow \infty} p(x, t) = P_S(x) \quad \text{for all initial distribution } p(x, 0)$$

- Unique stationary distribution
- Reached in long-time limit

Simulation:

- Parallel simulations/trajectories
 - long, sequential trajectory
- } see Chap 3



5.7 Stochastic calculus

GS Secs 13.7, 13.8, Jacobs Secs 3.4, 3.5

• SDE :

$$dX_t = F(X_t)dt + \sigma(X_t)dW_t$$

Hö SDE
infinitesimal
difference eq.

• Solution:

$$X_t = X_0 + \int_0^t F(X_s)ds + \int_0^t \sigma(X_s)dW_s$$

↑ random

- Stochastic integral
- Not "standard" Riemann integral
- Requires different calculus rules

• Example: $\int_0^T dW_t = W_T - W_0 = W_T$ • Example: $x(t)$ smooth function of t . Then

$$\int_0^T x(t)dx(t) = \frac{1}{2} \int_0^T dx(t)^2 = \frac{1}{2} x(T)^2$$

Standard
calculus• Example: $\int_0^T W_t dW_t \neq \frac{1}{2} W_T^2$

$$\text{In fact: } \int_0^T W_t dW_t = \frac{1}{2} (W_T^2 - T)$$

Stochastic
calculus

extra term

Jacobs
p. 33-• Example: $\int_0^T (dW_t)^2$

$$\text{Expectation: } E\left[\int_0^T (dW_t)^2\right] = \int_0^T E[(dW_t)^2] = \int_0^T dt = T$$

$$\text{Variance: } \text{var}\left(\int_0^T (dW_t)^2\right) = \text{var}\left(\sum_{i=0}^{n-1} (\Delta W)^2\right)$$

$$= \sum_{k=0}^{n-1} \text{var}(\Delta W)^2$$

$$= \sum_{i=0}^{n-1} 2 \Delta t^2$$

Variance of Gauss²
(χ^2 distribution)

$$= 2n \left(\frac{T}{n}\right)^2 = \frac{2T^2}{n} \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\Rightarrow \int_0^T (dW_t)^2 = T = \int_0^T dt$$

with probability 1

$$\Rightarrow dW_t^2 = dt$$

Hö's rule

• Itô's formula:

• SDE: $dX_t = F(X_t) dt + \sigma(X_t) dW_t$

• Transformation: $Z_t = f(X_t)$

Assumed smooth +
invertible

$$X_t = f^{-1}(Z_t)$$

• New SDE:

$$dZ_t = \left(\underbrace{F f'}_{\text{normal calculus}} + \underbrace{\frac{1}{2} \sigma^2 f''}_{\text{extra term Itô term}} \right) dt + f' \sigma dW_t$$

$$dZ_t = \underbrace{G(Z_t)}_{\text{new drift}} dt + \mu(Z_t) dW_t$$

• Note: Deterministic "standard" calculus term:

$$\begin{aligned} \dot{x}(t) &= F(x(t)) & \Rightarrow \dot{z}(t) &= f'(x(t)) \dot{x}(t) \\ z(t) &= f(x(t)) & &= f' F \end{aligned}$$

• "Proof" (idea behind formula):

$$Z_t = f(X_t)$$

$$\Rightarrow dZ_t = f'(X_t) dX_t + \frac{1}{2} f''(X_t) dX_t^2 + \dots$$

higher order

$$= f' (F dt + \sigma dW) + \frac{1}{2} f'' (F dt + \sigma dW)^2 + \dots$$

$$= f' F dt + f' \sigma dW + \frac{1}{2} f'' F^2 dt^2 + \frac{1}{2} f'' \sigma^2 dW^2$$

Itô term

higher order

$$+ \frac{1}{2} f'' 2 F \sigma dt dW$$

$$= \left(f' F + \frac{1}{2} f'' \sigma \right) dt + f' \sigma dW + \dots$$

higher order + ...

□

• Note: $\sigma = 0 \Rightarrow dZ_t = F f' dt$

Standard calculus

Example: $Z_t = W_t^2$

1- SDE: $dW_t = dW_t$ $F=0, \sigma=1$

2- Function: $Z_t = f(W_t)$ $f(w) = w^2$ $f' = 2w$ $f'' = 2$

$$\Rightarrow dZ_t = (Ff' + \frac{1}{2}\sigma^2 f'') dt + f' \sigma dW_t$$

$$= \underbrace{dt}_{\substack{\text{It\^o} \\ \text{term}}} + \underbrace{2W_t dW_t}_{\text{normal calculus}}$$

$$\Rightarrow Z_T = \underbrace{Z_0}_{=0} + \int_0^T dt + 2 \int_0^T W_t dW_t$$

$$= T + 2 \int_0^T W_t dW_t$$

But $dZ_t = dW_t^2$ $\therefore Z_T = W_T^2$

$$\Rightarrow \int_0^T W_t dW_t = \frac{W_T^2 - T}{2}$$

Example: $dY_t = \lambda Y_t dW_t$

$$\Rightarrow Y_t = e^{\lambda W_t - \frac{\lambda^2}{2} t}$$

Standard calculus would give

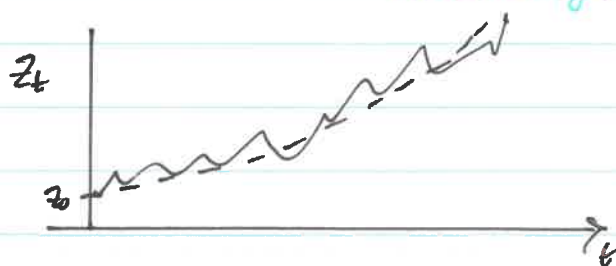
$$\frac{dY_t}{Y_t} = \lambda dW_t \quad \rightarrow \quad Y_t = Y_0 e^{\lambda W_t}$$

Example: $dZ_t = \mu Z_t dt + \sigma Z_t dW_t$

$$\Rightarrow Z_t = Z_0 e^{(\underbrace{\mu - \frac{\sigma^2}{2}}_{\text{corrected drift}})t + \sigma W_t}$$

Geometric BM

Model of
Stock prices



cf CWS

Example: $dX_t = \mu dt + \sigma dW_t$

$$Z_t = Z_0 e^{X_t}$$

To do. cf CWS

• Example: Ornstein-Uhlenbeck process

$$dX_t = -\gamma X_t dt + \sigma dW_t \quad \gamma, \sigma > 0$$

Take

$$Y_t = X_t e^{\gamma t} = f(X_t, t) \quad \rightarrow \text{added time dependence}$$

$$\begin{aligned} \Rightarrow dY_t &= \left(\partial_t f + F \partial_x f + \frac{\sigma^2}{2} \partial_{xx} f \right) dt + \partial_x f \sigma dW \\ &= \left(\underbrace{\gamma e^{\gamma t} X_t}_{Y_t} + \underbrace{(-\gamma X_t) e^{\gamma t}}_{Y_t} \right) dt + e^{\gamma t} \sigma dW \\ &= e^{\gamma t} \sigma dW_t \end{aligned}$$

$$\Rightarrow Y_t = Y_0 + \int_0^t e^{\gamma s} \sigma dW_s$$

$$\Rightarrow X_t = e^{-\gamma t} Y_t$$

$$= e^{-\gamma t} Y_0 + \sigma e^{-\gamma t} \int_0^t e^{\gamma s} dW_s$$

$$= e^{-\gamma t} X_0 + \sigma \int_0^t e^{\gamma(s-t)} dW_s$$

$$\begin{aligned} Y_0 &= X_0 e^0 \\ &= X_0 \end{aligned}$$

• Note: More examples in Jacobs p.53.

• Note: $\int_0^T f(X_t) dX_t$ gives different results depending

on discretization (Riemann) num used: left-pt, right-pt, etc.
In normal calculus, all discretizations give same result.

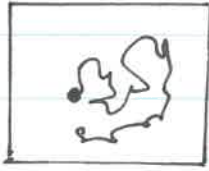
↪ Different calculus for integrals.

↪ Related to different rules for derivatives (Itô's formula)

See Higham's paper on num learn.

5.8 Physical Brownian motion

Jacobs Sec 5.1



- Particle (tiny) moving in fluid
- a = diameter (cross section)
- η = fluid viscosity

• Friction force: $F_{\text{friction}} = -\gamma p = -\gamma m v$

• Stokes law: $\gamma = \frac{6\pi\eta a}{m}$

• Newton's equation: $m \ddot{x} = F_{\text{friction}} + F_{\text{random}}$
 $= -\gamma m x + \sigma \xi(t)$

Gaussian white noise

$$\Rightarrow \frac{d}{dt} \begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} 0 & 1/m \\ 0 & -\gamma \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix} + \begin{pmatrix} 0 \\ \sigma \xi(t) \end{pmatrix}$$

or

$$\begin{pmatrix} dx_t \\ dp_t \end{pmatrix} = \begin{pmatrix} 0 & 1/m \\ 0 & -\gamma \end{pmatrix} \begin{pmatrix} x_t \\ p_t \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sigma \end{pmatrix} dW_t$$

2D linear equation

• Equation for momentum: $dp_t = -\gamma p_t dt + \sigma dW_t$

Ornstein-Uhlenbeck process

• Solution: $p_t = e^{-\gamma t} p_0 + \sigma \int_0^t e^{-\gamma(t-s)} dW_s$

- Verify with Itô's formula.

• Expectation: $E[p_t] = e^{-\gamma t} E[p_0] + \sigma \int_0^t e^{-\gamma(t-s)} \underbrace{E[dW_s]}_{=0}$

$$= e^{-\gamma t} E[p_0] \xrightarrow{t \rightarrow \infty} 0$$

• Variance: $\text{var}(p_t) = \sigma^2 \int_0^t e^{-2\gamma(t-s)} dt$

$$= \frac{\sigma^2}{2\gamma} (1 - e^{-2\gamma t})$$

$$\xrightarrow{t \rightarrow \infty} \frac{\sigma^2}{2\gamma}$$

Thermodynamic equipartition: Boltzmann's constant
 \sim Temperature

$$\langle E \rangle = \frac{\langle p^2 \rangle}{2m} = \frac{k_B T}{2} \times 2$$

$$\langle p^2 \rangle = \text{var}(P_{\infty})$$

2 dimension/degrees of freedom

$$\Rightarrow k_B T = \frac{\text{var}(P_{\infty})}{2m} = \frac{\sigma^2}{4m\gamma}$$

$$\Rightarrow \sigma = \sqrt{4m\gamma k_B T} = \sqrt{24\pi\eta a k_B T}$$

$$\Rightarrow \sigma \propto \sqrt{T}$$

- Thermal noise
- Fluctuation-dissipation theorem
- Dissipation (friction) related to noise amplitude (fluctuation)

Solution for X_t : $X_t = \int_0^t \frac{P_s}{m} ds$

$$E[X_t] = \frac{1}{m\gamma} (1 - e^{-\gamma t}) E[P_0] \rightarrow \frac{E[P_0]}{m\gamma} = 0$$

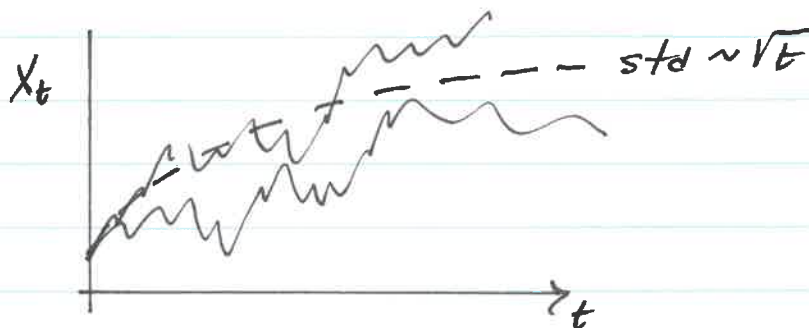
$$\text{var}(X_t) = \frac{\sigma^2 t}{(m\gamma)^2} + \frac{\sigma^2}{2m^2\gamma^3} (4e^{-\gamma t} - e^{-2\gamma t} - 3)$$

$$\sim \frac{\sigma^2 t}{(m\gamma)^2}$$

$$= \left(\frac{2k_B T}{3\pi\eta a} \right) t$$

$$= Dt$$

var \sim linear in time

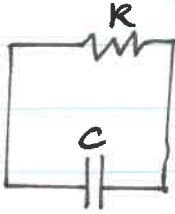


5.9 Johnson-Nyquist noise

Johnson, Thermal agitation of electricity, 1928 (experiment)

Nyquist, 1928 (theory)

Van Kampen p. 222



$$I = \frac{dQ_t}{dt}$$

$$C = \frac{Q_t}{V_t}$$

$$V_t = RI_t$$

$$I = \frac{V}{R} = \frac{Q}{RC}$$

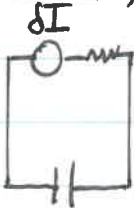
- Evolution equation: $\frac{dQ_t}{dt} = -\frac{Q_t}{RC} + \underbrace{\text{noise}}_{\delta I_t}$ Ornstein-Uhlenbeck equation
- Electrostatic energy: $\langle E \rangle = \frac{\langle Q^2 \rangle}{2C} = \frac{1}{2} k_B T$
- Fluctuating current: $\delta I_t = \sigma \xi_t$

$$\Rightarrow \sigma^2 = \frac{2k_B T}{R}, \quad \sigma = \sqrt{\frac{2k_B T}{R}}$$

$$\Rightarrow \sigma \propto \sqrt{T}$$

- Thermal electric noise
- Nyquist noise
- \sim nV at room temperature

• Current source



$$\delta I_t$$

• Voltage source



$$\delta V_t = R \delta I_t$$

5.10 Geometric BM and Finance

Jacobs Sec. 5.2

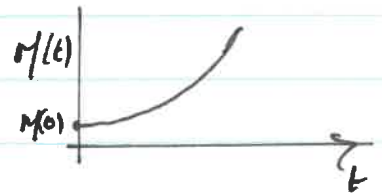
• Interest rate: M = amount of money

$$\Delta M = \underset{\substack{\uparrow \\ \text{interest rate} \\ \text{return}}}{k} M \underset{\text{Period}}{\Delta t}$$

$$\Rightarrow M(N\Delta t) = (1+k)^N M(0) \quad \text{compounded interest}$$

• Continuous interest: $dM = r M dt$

$$\Rightarrow M(t) = M(0) e^{rt} \quad \text{initial sum}$$



• Risk free: Money goes up deterministically,

• Stock market:

• S_t = price of some asset at time t

• "Mean" return: $dS_{\text{risk free}} = \mu S_t dt$

• Random variations: $dS_{\text{random}} = \sigma S_t dW_t$

volatility
Gaussian price movement
proportional to asset

$$\begin{aligned} \Rightarrow dS_t &= \mu S_t dt + \sigma S_t dW_t \\ &= (\mu dt + \sigma dW_t) S_t \end{aligned} \quad \text{drifted BM}$$

• Solution:

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}$$

