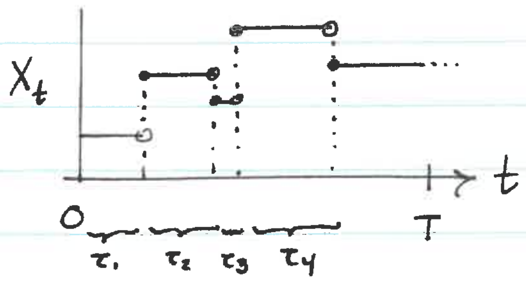
Chapter 3: Continuous-time Markov chains

GS Sec. 6.9



- Discrete space
- Continuous time
- Also called jump process

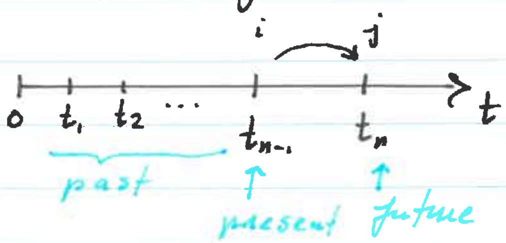
3.1 Definition

- Process: $\{X_t\}_{t=0}^{T}$ or $\{X_t : t \geq 0\}$ $\quad t \in \mathbb{R}^+$ time

- State space: $X_t \in \mathcal{X}$ assumed discrete (countable)
- Markov property:

$$P(X_{t_n} = j \mid X_{t_1} = i_1, \ldots, X_{t_{n-1}} = i_{n-1}) = P(X_{t_n} = j \mid X_{t_{n-1}} = i)$$

for all $i_1, \ldots, i_{n-1}, j$ and $0 \leq t_1 < \cdots < t_{n-1} < t_n$.



- Transition probability:

$$\pi_{ij}(s,t) = P(X_t = j \mid X_s = i) \qquad s \leq t$$
$$= P(X_s = i \rightarrow X_t = j)$$

- Homogeneous process: $\pi_{ij}(s,t)$ depends only on time difference $t-s > 0$ and not on absolute times s and t.

$$\pi_{ij}(s,t) = \pi_{ij}(0, t-s) \qquad \forall\, i,j$$

- Notation: $\pi_{ij}(t) = P(X_t = j \mid X_0 = i)$

$$= P(X_{t+s} = j \mid X_s = i)$$
$$= P(i \rightarrow j \text{ in time } t)$$

w7, h1

· Matrix form : $\Pi(t) = \left(\Pi_{ij}(t)\right)$

- $|\mathcal{X}| \times |\mathcal{X}|$ matrix
- Called : propagator
- $\Pi(0) = \mathbb{1}$
- $0 \le \Pi_{ij}(t) \le 1 \quad \forall i,j,t$
- $\sum_j \Pi_{ij}(t) = 1 \quad \forall t$

} Stochastic matrix

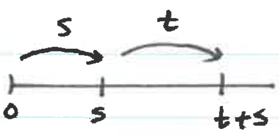
- $\Pi(s+t) = \Pi(s)\Pi(t) = \Pi(t)\Pi(s)$

Markov property
"Semi-group"

Proof :

$$\Pi_{ij}(s+t) = P(X_{s+t} = j \mid X_0 = i)$$

$$= \sum_k \underbrace{P(X_{s+t} = j \mid X_s = k)\, P(X_s = k \mid X_0 = i)}_{\text{matrix product}}$$



▨

Rem : Discrete time : $\Pi^2 = \Pi\Pi$ 2 time steps

Continuous time : $\Pi(2t) = \Pi(t)\Pi(t)$

· Probability vector : $p_j(t) = P(X_t = j)$

or $\bar{p}(t) = \left(p_1(t) \quad p_2(t) \quad \cdots \right)$ row vector

· Propagation equation :

$$p_j(t) = \sum_i p_i(0)\, \Pi_{ij}(t)$$

$$\bar{p}(t) = \bar{p}(0)\, \Pi(t)$$
$$= \bar{p}(0)\, \Pi(s)\, \Pi(t-s)$$
$$= \bar{p}(s)\, \Pi(t-s)$$

Chapman-Kolmogorov equation
or
"Master equation"

Rem: Discrete time

$$\bar{p}_n = \bar{p}_0\, \Pi^n$$
$$= \bar{p}_{n-1}\, \Pi$$

Continuous time

$$\bar{p}(t) = \bar{p}(0)\, \Pi(t)$$
$$= \bar{p}(s)\, \Pi(t-s)$$

W7, h1

3.2 Generator

· Infinitesimal propagator:

$$\underbrace{\Pi(\Delta t)}_{\text{matrix}} = \underbrace{\Pi(0)}_{\text{matrix}} + \underbrace{G}_{\text{matrix}}\Delta t + O(\Delta t^2)$$

$$= \mathbb{1} + G\Delta t + \cdots$$

· Generator: $\underbrace{G}_{\text{matrix}} = \lim_{\Delta t \to 0} \frac{\Pi(\Delta t) - \Pi(0)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Pi(\Delta t) - \mathbb{1}}{\Delta t}$

· off-diagonal elements:

$$\Pi_{ij}(\Delta t) = P(i \to j \text{ in } \Delta t) = G_{ij}\,\Delta t \qquad i \neq j$$

$$\Rightarrow \quad G_{ij} = \lim_{\Delta t} \frac{\Pi_{ij}(\Delta t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{P(i \to j \text{ in } \Delta t)}{\Delta t}$$

$$= W(i \to j) = W_{ij}$$

transition rate = probability per unit time
Real, positive numbers

· Diagonal elements:

$$\Pi_{ii}(\Delta t) = P(i \to i \text{ in } \Delta t)$$
$$= P(\text{stay in } i \text{ for } \Delta t)$$
$$= 1 + G_{ii}\,\Delta t$$

· Normalization:

$$1 = \sum_j \Pi_{ij}(\Delta t) = \Pi_{ii}(\Delta t) + \sum_{j \neq i} \Pi_{ij}(\Delta t)$$

$$= 1 + G_{ii}(\Delta t) + \sum_{j \neq i} G_{ij}\,\Delta t$$

$$\Rightarrow \quad G_{ii} + \sum_{j \neq i} G_{ij} = 0$$

· G_{ii} is not a free parameter. By convention:

$$G_{ii} = -\sum_{j\neq i} G_{ij} = -r_i$$

· Escape rate: $r_i = \sum_{j\neq i} G_{ij} = \sum_{j\neq i} W_{ij} = \sum_{j\neq i} W(i\to j)$

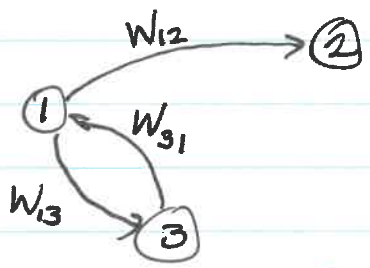
= total rate (prob/time) for leaving state i

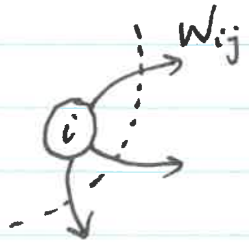
· Complete generator:

$$G_{ij} = \underbrace{W_{ij}}_{\substack{\text{off diagonal}\\ \text{jump rates}}} - \underbrace{r_i}_{\substack{\text{diagonal}\\ \text{escape rates}}}\delta_{ij}$$

$$\sum_j G_{ij} = 0 \iff G\bar{1} = \bar{0} \qquad G = \begin{pmatrix} \xrightarrow{\text{sum}} & \\ & \end{pmatrix} = 0$$

· Graphical representation:



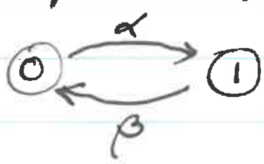


$r_i = \sum_{j\neq i} W_{ij}$

escape rate

No self loops!
Only transition rates shown

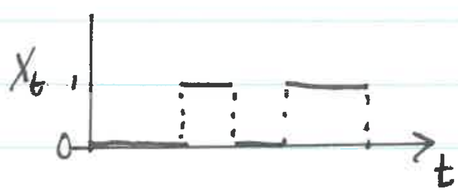
· Example: $X_t \in \{0,1\}$



$$W_{01} = \frac{P(0\to 1 \text{ in } \Delta t)}{\Delta t} = \alpha$$

$$W_{10} = \frac{P(1\to 0 \text{ in } \Delta t)}{\Delta t} = \beta$$

$$G = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$$



W7, h7, 2

3.3 Master equation

- Evolution over time t:

$$p_j(t) = \sum_i p_i(0)\,\Pi_{ij}(t) \qquad \bar{p}(t) = \bar{p}(0)\,\Pi(t)$$

- Evolution over interval Δt:

$$p_j(t+\Delta t) = \sum_i p_i(t)\,\Pi_{ij}(\Delta t) \qquad \bar{p}(t+\Delta t) = \bar{p}(t)\,\Pi(\Delta t)$$

$$= \sum_i p_i(t)\left(\delta_{ij} + G_{ij}\,\Delta t\right)$$

$$= p_j(t) + \sum_i G_{ij}\,p_i(t)\,\Delta t$$

$$\Rightarrow \quad \frac{d}{dt} p_j(t) = \lim_{\Delta t \to 0} \frac{p_j(t+\Delta t) - p_j(t)}{\Delta t} = \sum_i p_i(t)\,G_{ij}$$

i.e. $$\frac{d}{dt}\bar{p}(t) = \bar{p}(t)\,G$$

Master equation or Forward equation in GS

- Other form:

$$\frac{d}{dt} p_j(t) = \sum_i p_i(t)\,G_{ij}$$

$$= p_j(t)\underbrace{G_{jj}}_{-r_j} + \sum_{i\neq j} p_i(t)\,G_{ij}$$

$$= -\sum_{i\neq j} p_j(t)\,G_{ji} + \sum_{i\neq j} p_i(t)\,G_{ij}$$

$$\Rightarrow \quad \frac{d}{dt} p_j(t) = \sum_{i\neq j}\left[p_i(t)\,G_{ij} - G_{ji}\,p_j(t)\right]$$

$$= \sum_{i\neq j}\Big[\underbrace{p_i(t)\,W_{ij}}_{\substack{\text{flow } i\to j \\ \text{in}}} - \underbrace{W_{ji}\,p_j(t)}_{\substack{\text{flow } j \text{ to } i \\ \text{out}}}\Big]$$

1v7, h2

· Solution : $\bar{p}(t) = \bar{p}(0)\, e^{Gt}$

$= \bar{p}(0)\, \underbrace{\Pi(t)}_{\text{matrix}} \quad \Rightarrow \quad \Pi(t) = e^{Gt}$

3.4 Ergodic Markov chains

See notes on Markov chains Sec. 2.5

· Stationary distribution : Time-independent distribution $\bar{p}^*$ such that

$$\frac{d}{dt}\bar{p}^* = 0 \quad \Longleftrightarrow \quad \bar{p}^* G = 0$$

- Eigenvector (left) of G with eigenvalue 0
- $\bar{p}^* \Pi(t) = \bar{p}^* \quad \forall\ t$
- Eigenvector of $\Pi(t)$ with eigenvalue 1

· Limiting distribution : $\bar{p}(\infty) = \lim_{t\to\infty} \bar{p}(t) = \lim_{t\to\infty} \bar{p}(0)\Pi(t)$

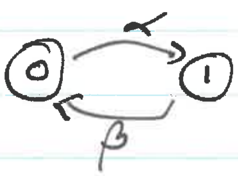
- Doesn't necessarily exist
- Can depend on choice of initial dist. $\bar{p}(0)$
- Interesting case : $\bar{p}(\infty)$ exist + ind. of $\bar{p}(0)$

· Ergodic Markov chain (continuous time) : $\{X_t\}_{t=0}^{T}$ is aperiodic and irreducible. Then

$$\bar{p}(\infty) = \lim_{t\to\infty} p(0)\,\Pi(t) = \bar{p}^* \qquad \forall\ \bar{p}(0)$$

· Example : $X_t \in \{0,1\}$

$$G = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix} \qquad \bar{p}^* = \begin{pmatrix} \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{pmatrix}$$

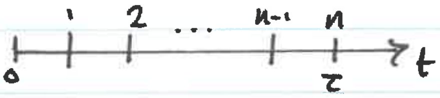


$0 < \alpha, \beta < \infty$

Why?

W7.L2

3.5 Residence / sojourn time



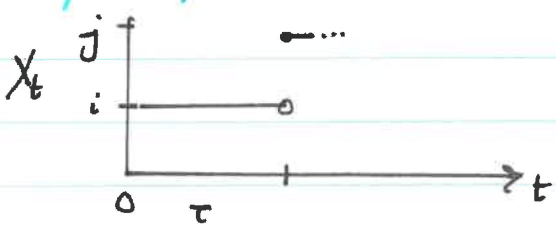
$$n = \frac{\tau}{\Delta t}$$

$$\begin{aligned} P(X_t = i : t \in [0,\tau)) &= P(X_t \text{ stays in } i \text{ for time } \tau) \\ &= P(\text{no jumps in } \tau) \\ &= P(\text{no jump in } \Delta t)^n \\ &= \Pi_{ii}(\Delta t)^{\tau/\Delta t} \\ &= \left(1 - r_{ii}\frac{\tau}{n}\right)^n \xrightarrow{\Delta t \to 0} e^{-r_i \tau} \end{aligned}$$

- Residence / sojourn time in state $i \sim \text{Exp}(r_i)$ Holding time

3.6 Jump representation

GS p. 259



1- Start at $X_t = i$

2- Stay in i for random time $\tau \sim \text{Exp}(r_i)$

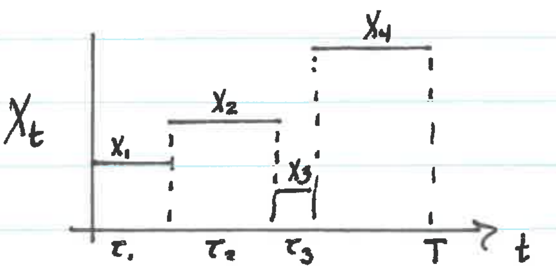
3- After τ, jump to $j \neq i$ with probability

$$P(i \to j) = \frac{W(i \to j)}{r_i}$$

one jump, "instant"

4- Repeat

- Note : $\sum_j P(i \to j) = \sum_j \frac{W(i \to j)}{r_i} = \frac{r_i}{r_i} = 1 \quad \forall i.$ as expected



$$\{X_t\}_{t=0}^{T}$$

$$\downarrow$$

$\{X_1, X_2, X_3, \dots\}$ states visited

$\{\tau_1, \tau_2, \tau_3, \dots\}$ Sojourn times

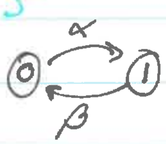
W9, h1

3.7 Examples

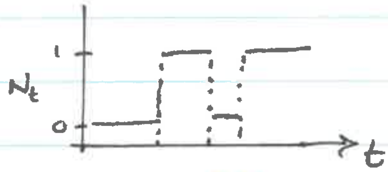
3.7.1 Bernoulli process (telegraph noise)

Jacobs Sec 8.5

$N_t \in \{0, 1\}$



$$G = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$$



Morse code " telegraph

- Master equation:

$$\dot{P}(0,t) = \beta P(1,t) - \alpha P(0,t)$$
$$\dot{P}(1,t) = \underbrace{\alpha P(0,t)}_{\substack{\text{in flow} \\ \text{off diagonal}}} - \underbrace{\beta P(1,t)}_{\substack{\text{out flow} \\ \text{diagonal}}}$$

- Solution:

$$P(n,t) = e^{-\gamma t} P(n,0) + \frac{\mu_n}{\gamma}(1 - e^{-\gamma t})$$

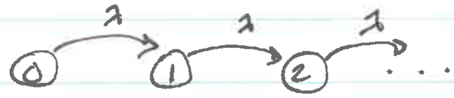
$$\mu_1 = \alpha \qquad \mu_0 = \beta \qquad \gamma = \mu_1 + \mu_0$$

- Stationary distribution: $P^*(n) = \mu_n / \gamma$

$$P^*(0) = \frac{\beta}{\alpha+\beta}$$
$$P^*(1) = \frac{\alpha}{\alpha+\beta}$$

3.7.2 Poisson process

GS 6.8, Jacobs 8.1



- $N_t \in \{0, 1, 2, \dots\}$
- $N_0 = 0$ by convention
- $P(n,t) = P(N_t = n)$
- Master equation: $\dot{P}(n,t) = \underbrace{\lambda P(n-1,t)}_{\text{in}} - \underbrace{\lambda P(n,t)}_{\text{out}}$

$$P(n,0) = \delta_{n,0} \qquad N_0 = 0$$

- Solution: $P(0,0) = 1$

$$P(0,t) = e^{-\lambda t} \qquad \text{residence time at } 0$$
$$\vdots$$
$$P(n,t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

Poisson distribution with parameter λt

- Expectation: $E[N_t] = \lambda t$

Counting process

Proof 1: Direct substitution.
Proof 2: GS p. 248 based on arrival times.
Proof 3: Discretized time:

$N = \dfrac{t}{\Delta t}$

$P(+1 \text{ jump in } \Delta t) = \lambda \Delta t = p$

$P(\text{no jump in } \Delta t) = 1 - \lambda\Delta t = 1 - p$

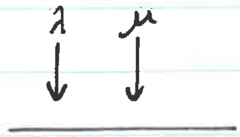
$\{N_t = n\} = \{n \text{ 1's in } N \text{ steps}\}$
$\qquad\quad = \{N-n \text{ 0's in } N \text{ steps}\}$ } Binomial!

$\Rightarrow P(N_t = n) = \lim\limits_{\substack{\Delta t \to 0 \\ N \to \infty}} \text{Bin}(N, \lambda\Delta t) = \text{Poisson}(\lambda t)$

See CW1 ☒

3.7.3 Superposition of Poisson processes

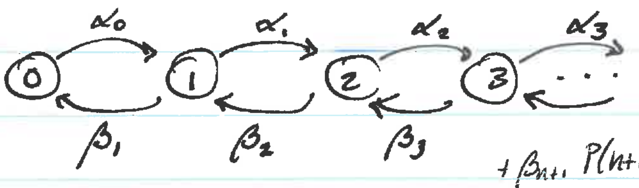
- X_t Poisson process intensity λ
- Y_t " " " μ
- $X_t \perp\!\!\!\perp Y_t$



$\Rightarrow Z_t = X_t + Y_t$ is Poisson process with intensity $\lambda + \mu$

See CW3

3.7.4 Birth-death process



$+\beta_{n+1} P(n+1, t) \qquad - \alpha_n P(n,t)$

· Master equation: $\dot{P}(n,t) = \underbrace{\alpha_{n-1} P(n-1,t)}_{\text{in}} - \underbrace{\beta_n P(n,t)}_{\text{out}}$

· Standard model: $\alpha_n = \alpha n$ compounded rates:
$\beta_n = \beta n$ n individuals with rate α, β each $\Rightarrow$ total rates $n\alpha, n\beta$

· Pure birth (Yule-Furry model): $\alpha_n = \alpha n$ Compounded
$\beta_n = 0$ Poisson process

· Example : Predator - Prey model see CW3

Jacobs 8.8

· Prey : M Mice

· Predator : J Jaguar

· "Jumps" :

(1) $M \to M+1$ rate λM mice birth

(2) $J \to J+1$, $M \to M-1$ rate $\mu J M$ mouse eaten jaguar birth

(3) $J \to J-1$ rate νJ jaguar death

· State : (M_t, J_t)

· Master equation :

$$\dot{P}(j,m,t) = \lambda(m-1)\,P(j,m-1,t) \quad (1)$$

$$+\,\mu\,(j-1)(m+1)\,P(j-1,m+1,t) \quad (2)$$

$$+\,\nu\,(j+1)\,P(j+1,m,t) \quad (3)$$

$$-\,\lambda m\,P(j,m,t)$$
$$-\,\mu j m\,P(j,m,t)$$
$$-\,\nu j\,P(j,m,t)$$

diagonal terms

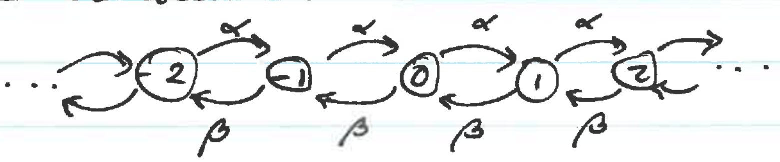
· Expectation :

$$\dot{\bar{m}} = \lambda\bar{m} - \mu\bar{j}\bar{m} \qquad \bar{m} = E[M_t]$$

$$\dot{\bar{j}} = \mu\bar{j}\bar{m} - \nu\bar{j} \qquad \bar{j} = E[J_t]$$

Lotka-Volterra equations !

3.7.5 Random Walk



· Master equation: $\dot{P}(n,t) = \underbrace{\alpha P(n-1,t) + \beta P(n+1,t)}_{\text{in}}$

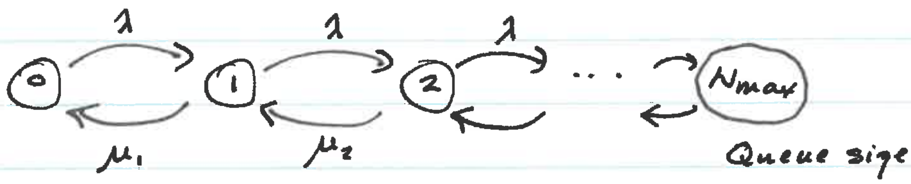
$$\underbrace{-(\alpha+\beta)\,P(n,t)}_{\text{out}}$$

· Generator :

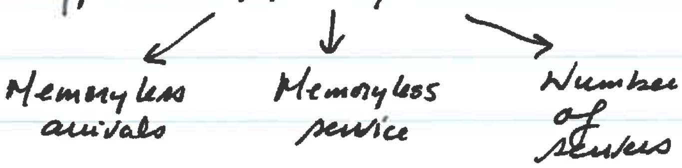
$$G = \begin{pmatrix} \ddots & & & & 0 \\ \ddots & \beta & -\alpha-\beta & \alpha & \\ 0 & & & \ddots & \ddots \end{pmatrix}$$

More in Chapter 4

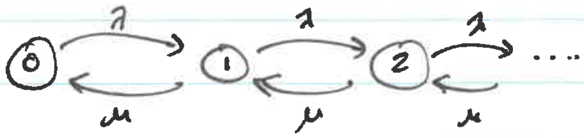
3.7.6 Queues



- Poisson arrivals : $\lambda_i = \lambda$ no compounding, why?
- Poisson service : μ_i
- Queue type : M / M / n

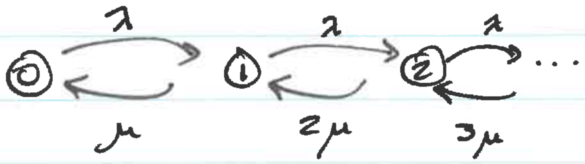


- M/M/1 : $\lambda_i = \lambda$, $\mu_i = \mu$



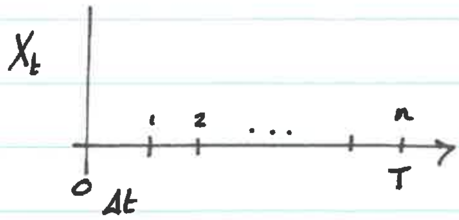
- M/M/∞ : $\lambda_i = \lambda$

  $\mu_i = i\mu$ every arrival served



3.8 Simulations

3.8.1 Discretized time method



$$n = \frac{T}{\Delta t}$$

$$\{X_t\}_{t=0}^{T} \xrightarrow[\Delta t]{\text{discretization}} \{X_i\}_{i=0}^{n}$$

continuous time — discrete time

· Transition probability:

$$P(i \to j \text{ in } \Delta t) = \Pi_{ij}(\Delta t)$$
$$= \delta_{ij} + G_{ij}\,\Delta t$$

$$\Rightarrow P(i \to j \text{ in } \Delta t) = W_{ij}\,\Delta t \qquad i \neq j$$

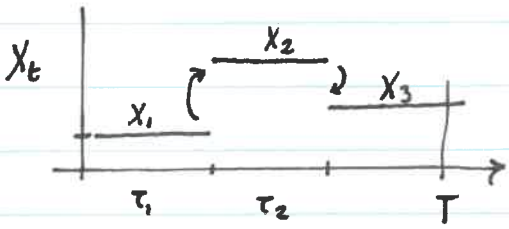
· Algorithm/method:

1. Start $X_0 = i$ random or deterministic
2. Attempt jump to $j \neq i$ with probability $W_{ij}\,\Delta t$
3. Update:
$$X_{\Delta t} = \begin{cases} j & \text{jump accepted} \\ i & \text{" rejected} \end{cases}$$
4. Repeat

↪ See notes on discrete-time Markov chains for pseudo code

· Rem: Δt must be chosen small enough so that $W_{ij}\,\Delta t < 1$ over all i, j. Why?

3.8.2 Random jump time method



$\{X_t\}_{t=0}^{T} \longrightarrow$ $\{X_1, X_2, \dots\}$ states visited

$\{\tau_1, \tau_2, \dots\}$ holding / sojourn times

· Sojourn times: $\tau_i \sim \mathrm{Exp}(r_i)$ $\quad r_i = \sum_{j\neq i} W_{ij}$ escape rate

· Jump probabilities: $P(i\to j) = \dfrac{W_{ij}}{r_i}$ instantaneous jump

· Algorithm/method:

1. Start $X_0 = i \longrightarrow X_1$
2. Draw $\tau_1 \sim \mathrm{Exp}(r_i) \longrightarrow \tau_1$
3. Attempt jump to $j \neq i$ with probability W_{ij}/r_i
4. Update:

   $X_{\tau_1} = j$ so $X_t = i \;\; \forall\, t \in [0, \tau_1)$

5. Repeat

· Rem: X_t = constant between jumps.

Summary

Discrete time	Continuous time
$\{X_i\}_{i=1}^{n} \quad i \in \mathbb{N}$	$\{X_t\}_{t=0}^{T} \quad t \in \mathbb{R}$
$p_n(i) = P(X_n = i)$	$p_i(t) = P(X_t = i)$
$\Pi_{ij} = P(i \to j) = P(X_n = j \mid X_{n-1} = i)$	$\Pi_{ij}(t) = P(i \to j \text{ in } t) = P(X_t = j \mid X_0 = i)$
$\bar{p}_n = \bar{p}_{n-1}\,\Pi = \bar{p}_0\,\Pi^n$	$\bar{p}(t) = \bar{p}(s)\,\Pi(t-s) = \bar{p}(0)\,\Pi(t)$
$\bar{p}^* = \bar{p}^*\,\Pi$	$\bar{p}^* = \bar{p}^*\,\Pi(t) \quad \forall t$

$$\Pi(t) = e^{Gt}$$
$$\Pi(0) = \mathbb{1}$$

$$\Pi(\Delta t) = \mathbb{1} + G\,\Delta t$$

$$G_{ij} = W_{ij} - r_i\,\delta_{ij}$$

$$W_{ij} = W(i \to j) = \lim_{\Delta t} \frac{P(i \to j \text{ in } \Delta t)}{\Delta t}$$

$$r_i = \sum_j W_{ij}$$

$$\bar{p}^* G = \bar{0}$$