Department of Mathematical Sciences
Division of Applied Mathematics
Stellenbosch University

2019 | MS763 | Applied Markov Processes

Problems for Chapter 4: Brownian motion

Due: 11 October 2019

Theoretical

Q1. (Diffusion equation) Show that the time-dependent Gaussian kernel seen in class solves the diffusion equation:

$$\frac{\partial}{\partial t} p(x,t) = \frac{D}{2} \frac{\partial^2}{\partial x^2} p(x,t). \tag{1}$$

[Note for the physicists: This is also the equation describing the diffusion of heat in solids. Why?]

Q2. (Biased diffusion) Consider a variation of the Bernoulli random walk seen in class in which the jump random variable is biased such that $P(Y = 1) = \alpha$, $P(Y = -1) = \beta = 1 - \alpha$.

(a) Find the corresponding diffusion equation using the continuum limit, as seen in class.

(b) What is the time-dependent solution $p(x,t)$ of this new diffusion equation? Assume $X_0 = 0$.

Numerical

Q3. (Simple random walk)

(a) Generate and plot 100 sample paths of the Bernoulli random walk. Plot the random paths up to X_{20} using $X_0 = 0$ and $p = 0.3$.

(b) Construct a histogram of the position reached at time $n = 5$. Compare your result with the corresponding binomial distribution with bias p.

(c) Analyse your results. What happens if you change p?

Q4. (Brownian motion)

(a) Generate and plot 50 sample paths of the Wiener process W_t for $t \in [0, 1]$. Use $\Delta t = 0.1$. Then repeat with $\Delta t = 0.01$ and $\Delta t = 0.005$. Analyse your results. Are the sample paths differentiable?

(b) Write a code to verify that the standard deviation of W_t grows in time like $\sqrt{t}$. No need to put error bars, but analyse your results.

(c) Write a code to verify that the distribution of W_t follows the Gaussian probability kernel seen in class. Analyse your results.

Q5. (Brownian motion with reset) Consider a Brownian motion X_t in one dimension which is "reset" to the origin at independent and exponentially-distributed random times. This is an example of mixed diffusion-jump process which can be simulated in the following way using the discretized-time method: at each time step Δt, jump to 0 with probability $\lambda \Delta t$ or perform a Brownian motion step with complementary probability $1 - \lambda \Delta t$.

(a) Modify your code of Q4 to generate and plot one sample path of the reset Brownian motion for $t \in [0, T]$. Use $\lambda = 1$ and $T = 2$.

(b) Estimate numerically the expected number of resets as a function of the integration time T.

(c) Estimate numerically (with a histogram) the stationary distribution of Brownian motion with reset. Can you fit the result with a known function?

Q6. (Planar Brownian motion) Generate and plot trajectories of Brownian motion in two dimensions. Do not plot the trajectories as a function of time, but directly in the (x, y) plane.

Reading

- Brownian motion on Wikipedia
- Johnson–Nyquist noise on Wikipedia

Prize question

R100 for the best complete answer. Hand in your solution on a separate sheet.

What is the modified diffusion equation describing the evolution of $p(x,t)$ for the Brownian motion with reset described in Q5? Solve this equation to find the stationary distribution of that process.