7.1. Large deviations for jump processes

· Process : $X_t \in \Lambda = \underbrace{\{1,2,3,\dots,9\}}_{\text{discrete set of states}}$

· Transition rates : $W_{ji} = W(i \to j)$ ~ Prob/time jump $i \to j$

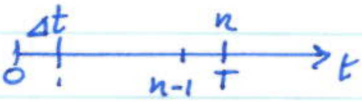
· Escape rate : $r_i = \sum_{j \neq i} W_{ji} := -G_{ii}$ ~ escape rate from i

· Full generator : $G_{ji} = \underbrace{W_{ji}}_{\text{off diagonal}} - r_i \underbrace{\delta_{ij}}_{\text{diagonal}}$

· Sample mean : $S_T = \frac{1}{T}\int_0^T f(X_t)\, dt$

→ Continuous-time analog of $S_n = \frac{1}{n}\sum_i f(X_i)$

· Problem : ① LDP for S_T ?
② Rate function ?

Discretization approach:  $X_i = X_{i\Delta t}$

① Discretization of S_T :

$$S_T = \frac{1}{n\Delta t}\sum_{i=1}^{n} f(X_i)\,\Delta t = S_n$$

② Process X_t → Markov chain X_i

· Infinitesimal propagator: $\Pi_{\Delta t} = e^{G\Delta t} = \mathbb{1} + G\Delta t$

· Transition probabilities:

$$\begin{aligned}\Pi_{\Delta t}(j|i) &= P(i \to j \text{ in } \Delta t) \qquad j \neq i\\ &= G_{ji}\ \Delta t\\ &= W_{ji}\ \Delta t\end{aligned}$$

$$\begin{aligned}\Pi_{\Delta t}(i|i) &= \cancel{0}\\ &= 1 - r_i \Delta t\end{aligned}$$

no more than one transition in Δt

(3) SCGF :

$$\lambda(k) = \lim_{T\to\infty} \frac{1}{T} \ln E[e^{TkS_T}]$$

$$\downarrow$$

$$\lambda(k) = \lim_{\substack{n\to\infty \\ \Delta t\to 0 \\ n\Delta t\to\infty}} \frac{1}{n\Delta t} \ln E[e^{n\Delta t\, k S_n}]$$

$$= \lim_{n\Delta t\to\infty} \frac{1}{n\Delta t} \ln E[e^{\Delta t\, k \sum_i f(X_i)}]$$

Results of week 5 : Tilted matrix :

$$\Pi_{k,\Delta t}(j|i) = e^{k\Delta t f(j)}\,\Pi_{\Delta t}(j|i)$$

$$= (1 + k\Delta t f(j))(\mathbb{1} + G_{ji}\Delta t) \qquad \Delta t\to 0$$
$$= \mathbb{1} + (G_k)_{ji}\,\Delta t$$

· Tilted generator : $G_k = G + \underbrace{k f(j)\mathbb{1}}_{\text{diagonal}}$

that is,

$$(G_k)_{ji} = G_{ji} + k f(j)\,\delta_{ij}$$
$$= G_{ji} + k f(i)\,\delta_{ij}$$

$$= W_{ji} + (\underbrace{k f(i) - r(i)}_{\text{modified escape rate}})\,\delta_{ij}$$

· Dominant eigenvalue of G_k : $\zeta(G_k)$

· Dominant eigenvalue of $\Pi_{\Delta t,k}$: $\zeta(\Pi_{k,\Delta t}) = e^{\Delta t\,\zeta(G_k)}$

$$\Rightarrow \lambda(k) \sim \frac{1}{n\Delta t} \ln e^{\Delta t\,\zeta(G_k)\,n} = \zeta(G_k)$$

$$\left[\ \lambda(k) = \zeta(G_k)\ \right]$$

④ Gärtner-Ellis Theorem:

- $\lambda(k)$ differentiable? $\Rightarrow$ ① LDP for S_T
  ② $I(s) = \sup_k \{ks - \lambda(k)\}$

Rem: Other approach: Feynman-Kac integrals / pde for $E[e^{TkA_T}]$.

Example: (Exercise 3.6.11 of HT2011)

$$\begin{tikzcd} ⓪ \arrow[r, bend left, "\alpha"] & ① \arrow[l, bend left, "\beta"] \end{tikzcd} \qquad G = \begin{pmatrix} -\alpha & \beta \\ \alpha & -\beta \end{pmatrix}$$

i.e. $W_{10} = W(0 \to 1) = \alpha$
$W_{01} = W(1 \to 0) = \beta$

- $L_{T,0} = \frac{1}{T}\int_0^T \delta_{X_t,0}\, dt$ = mean occupation time at 0

- $f(i) = \begin{cases} 1 & i=0 \\ 0 & i=1 \end{cases}$

- Tilted generator:

$$(G_k)_{ji} = G_{ji} + k f(i)\delta_{ij}$$

$$G_k = \begin{pmatrix} -\alpha & \beta \\ \alpha & -\beta \end{pmatrix} + k\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

modified rate at 0

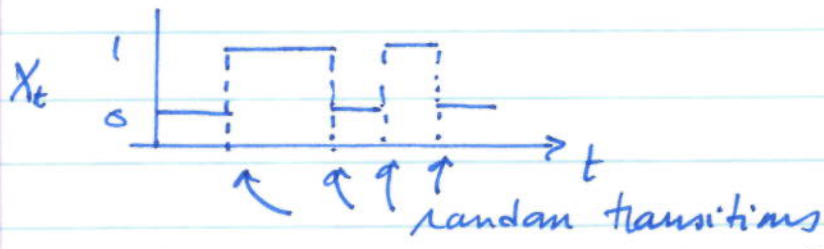
$$= \begin{pmatrix} -\alpha + k & \beta \\ \alpha & -\beta \end{pmatrix}$$

Matrix is not stochastic anymore

$\rightarrow$ Find $\zeta(G_k)$
Find $I(l)$ for $P(L_{T,0} = l)$ $\quad l \in [0,1]$

$\rightarrow$ Exercise for this week.

7.2. Pair (current-type) observable



· Consider observable that is dependent on these transitions:

$$S_n = \frac{1}{n} \sum_{\text{jumps}} g(X_{t^-}, X_{t^+})$$

state before jump — state after jump

$$= \frac{1}{n} \sum_{t\,:\,\Delta X_t \neq 0} g(X_{t^-}, X_{t^+})$$

sum over all jumps $\Delta X_t = X_{t+} - X_{t-} \neq 0$

or

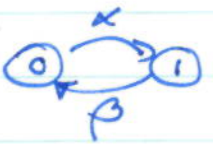
$$S_T = \frac{1}{T} \sum_{t\,:\,\Delta X_t \neq 0} g(X_{t^-}, X_{t^+})$$

· <u>Result</u>: $\lambda(k) = \zeta(G_k)$

$$(G_k)_{ji} = G_{ji}\, e^{k g(i,j)}$$

→ Derivation as an exercise.

Example: (Exercise 3.6.12 of HT2011) cf list of errata at end of HT2011
3.6.13



$$Q_T = \frac{1}{T} \sum_{t\,:\,\Delta X_t \neq 0} g(X_{t-}, X_{t+})$$

$$g(x,y) = 1 - \delta_{x,y} = \begin{cases} 0 & x = y \quad \text{not jump} \\ 1 & x \neq y \quad \text{jump} \end{cases}$$

$$Q_T = \frac{1}{T}\ \text{Number of random jumps in } [0,T]$$

→ Left as exercise.

7.3. Large deviations for stochastic differential equations

Particular case: 1D SDE on $\mathbb{R}$:

$$dX_t = \underbrace{F(X_t)\,dt}_{\text{deterministic part/drift}} + \underbrace{\sigma\, dW_t}_{\text{Wiener motion/diffusion of Week 6}}$$

· Generator: $L = F(x)\frac{d}{dx} + \frac{\sigma^2}{2}\frac{d^2}{dx^2}$

· Sample mean: $S_T = \frac{1}{T}\int_0^T f(X_t)\,dt$

· Result: SCGF: $\lambda(k) = \zeta(L_k)$

Tilted generator: $L_k = L + kf(x)$

$$= F(x)\frac{d}{dx} + \frac{\sigma^2}{2}\frac{d^2}{dx^2} + kf(x)$$

→ Result not proved in class.

Methods of proof: ① Discretization
or ② Feynman-Kac formula.

· Dominant eigenvalue:

$$\underbrace{L_k}_{\text{in general not hermitian!}} \underbrace{\varphi(x)}_{} = \lambda(k)\, \underbrace{\varphi(x)}_{\text{eigenfunction}}$$

Spectral problem

Example: Langevin or Ornstein-Uhlenbeck process:

$$dX_t = \underbrace{-aX_t\,dt + \sigma\, dW_t}_{\text{linear SDE}}$$

$$S_T = \frac{1}{T}\int_0^T X_t\,dt$$

$$L_k = -ax\frac{d}{dx} + \frac{\sigma^2}{2}\frac{d^2}{dx^2} + kx$$ not self adjoint

→ Exercise: Find $\lambda(k)$

Find $I(s)$

Answer: $I(s) = \dfrac{a^2 s^2}{2\sigma^2}$ Gaussian fluctuations!

Other observable:

$$S_T = \frac{1}{T}\int_0^T X_t^2\,dt$$

$$L_k = -ax\frac{d}{dx} + \frac{\sigma^2}{2}\frac{d^2}{dx^2} + kx^2$$

→ Exercise: Find $\lambda(k)$

" $I(s)$

Answer: ...

7.4. Sanov result for Markov processes

- Markov process: X_t
- Empirical distribution:

$$L_{T,i} = \frac{1}{T}\int_0^T \delta_{X_t,i}\, dt \qquad \text{discrete case}$$

$$L_T(x) = \frac{1}{T}\int_0^T \delta(X_t - x)\, dt \qquad \text{continuous case}$$

= fraction of time spent in i or in x

- LDP: $P(L_T = \rho) \asymp e^{-T I(\rho)} \qquad T \to \infty$
- Rate function:

$$I(\rho) = -\inf_{v>0} \left\langle \ln \frac{Gv}{v} \right\rangle_\rho$$

$$= -\inf_{v>0} \int dx\, \rho(x) \ln \frac{(Gv)(x)}{v(x)} \qquad \text{not explicit}$$

- Particular case: X_t is reversible (detailed balance). Then

$$I(\rho) = -\left\langle \sqrt{\frac{\rho}{p_s}},\, G\sqrt{\frac{\rho}{p_s}} \right\rangle_{p_s} \qquad \text{Dirichlet form } \geq 0$$

p_s: stationary/invariant density

apply G on $\sqrt{\frac{\rho}{p_s}}$

$$I(\rho) = -\int dx\, p_s(x) \sqrt{\frac{\rho(x)}{p_s(x)}} \left(G\sqrt{\frac{\rho}{p_s}}\right)(x)$$

Example: O-U process: $G = L$.

→ Exercise

Answer: $I(\rho) = \frac{\sigma^2}{2}\int dx\, p_s(x) \underbrace{\left(\frac{d}{dx}\sqrt{\frac{\rho(x)}{p_s(x)}}\right)^2}_{\geq 0}$

Zero of $I(\rho)$?