Information and control:
A tale of statistical physics and engineering

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Outline

H. Touchette (with Seth Lloyd)
Information-theoretic aspects in the control of dynamical systems
M.Sc. Thesis, MIT, 2000
Phys. Rev. Lett. 84, 1156, 2000
Physica A 331, 140-172, 2004

Landscape

Control theory

- Steam engines (Watts 1776)
- Mechanics
- Dynamical systems (Lyapunov 1892)
- Stochastic systems
- Optimal control theory
- Dynamic programming (Bellman 1957)

Information theory

- Norbert Wiener 1948
- Claude Shannon 1948
- Probability theory
- Stochastic systems

Physics

- Classical mechanics
- Quantum control
- Statistical mechanics

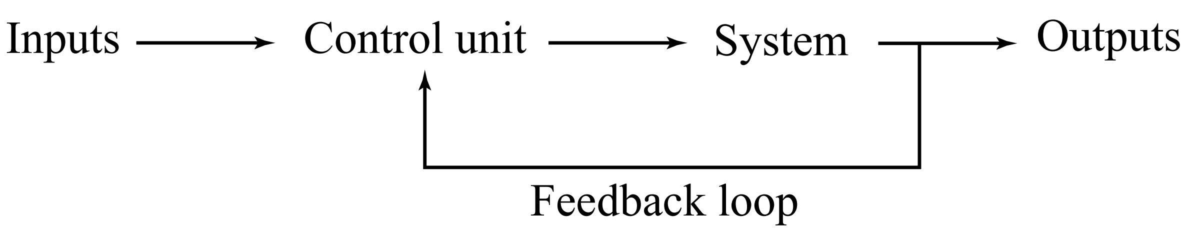
- Control: Optimize some function with given set of actions
- Strong link with optimization theory

Control systems

Examples

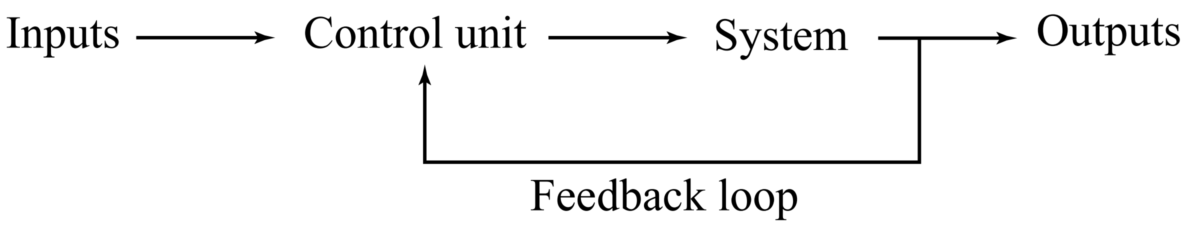
- Human drivers / auto-pilot systems
- Cars: fuel injection, anti-skidding, anti-lock breaks,...
- Heating / cooling systems

controller = sensor + actuator



- **Sensor**: What sees the system to control
- **Actuator**: What acts on the system
- **Design** or **protocol**: Given sensing-action sequence

Open- vs closed-loop control



Open-loop control

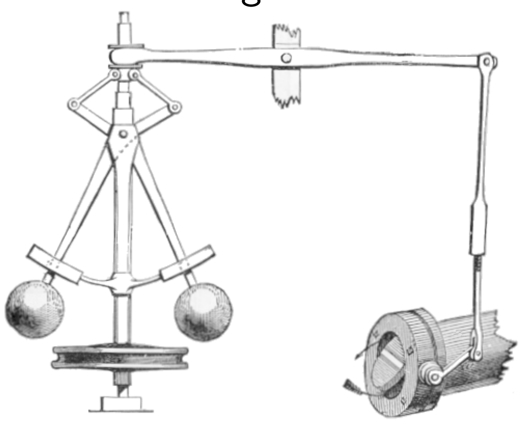
No sensor – no information required

Closed-loop or **feedback** control

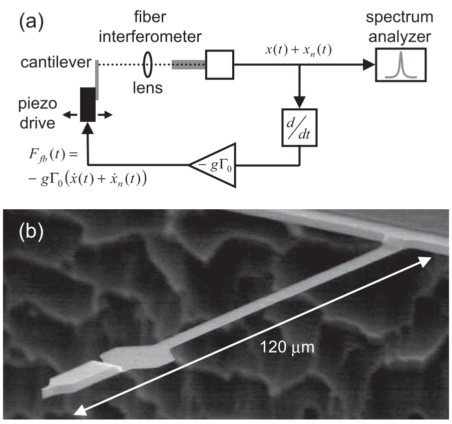
With sensor – information required and used

Examples

Watts governor

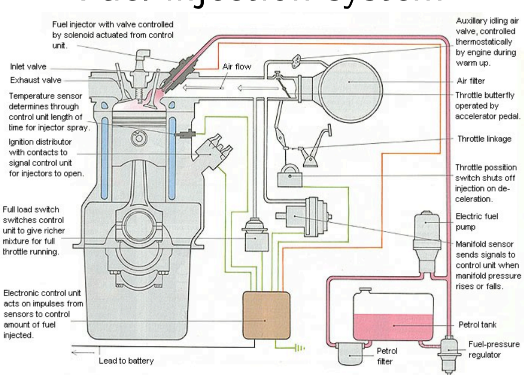


Cantilever cooling

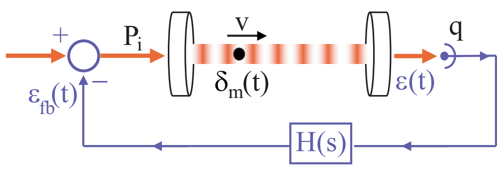


(Poggio et al PRL 2007)

Fuel injection system



Molecular cooling



(Velutic et al PRE 2007)

Problem

"the problems of control engineering and communication engineering were inseparables, and that they centered around the fundamental notion of the message"
– Norbert Wiener, 1948

Questions

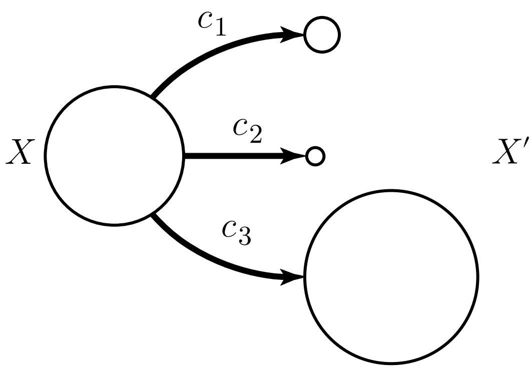
- How much information is used in control?
- How to define information?
- Is there a trade-off information/performance?

Approach

- Control reduces uncertainty/variability (Wiener, Ashby)
- Uncertainty/variability = entropy (Maxwell, Shannon)
- Information = mutual information (Shannon)

Control as entropy reduction

Initial state



Target state

Good control

- Target state reached from many initial states
- Entropy is reduced

Bad control

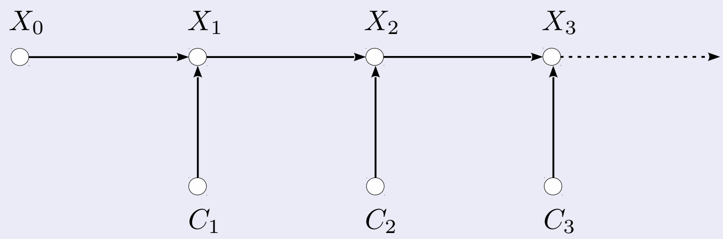
- Control randomizes the initial state
- Entropy is increased
- (Sometimes good: Anti-control, mixing, etc.)

Control diagrams

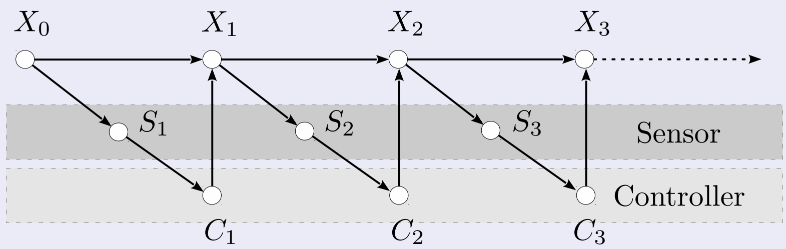
(Neapolitan 1989; Pearl 2009)

- Causal graphs in discrete time
- Directed acyclic graphs (DAGs)

Open-loop control

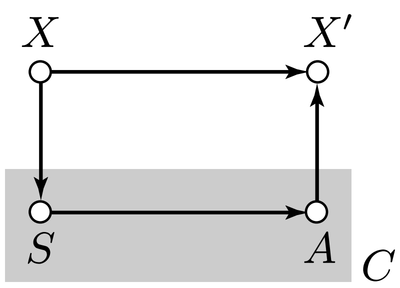


Closed-loop control



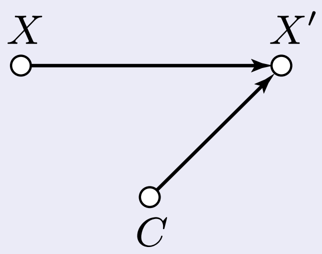
Basic DAG

(HT & Lloyd PRL 2000, Physica A 2004)



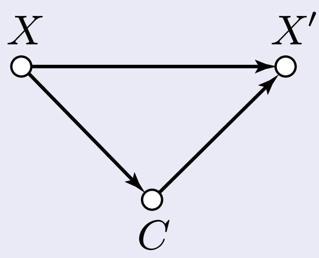
- X: controlled system
- S: sensor
- A: actuator

Open-loop



$$p(x, x', c) = p(x)p(c)p(x'|x, c)$$

Closed-loop



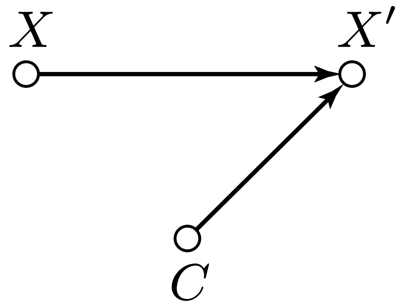
$$p(x, x', c) = p(x)p(c|x)p(x'|x, c)$$

Open-loop control

Entropy reduction

$$\Delta H_{\text{open}} = H(X) - H(X')_{\text{open}}$$

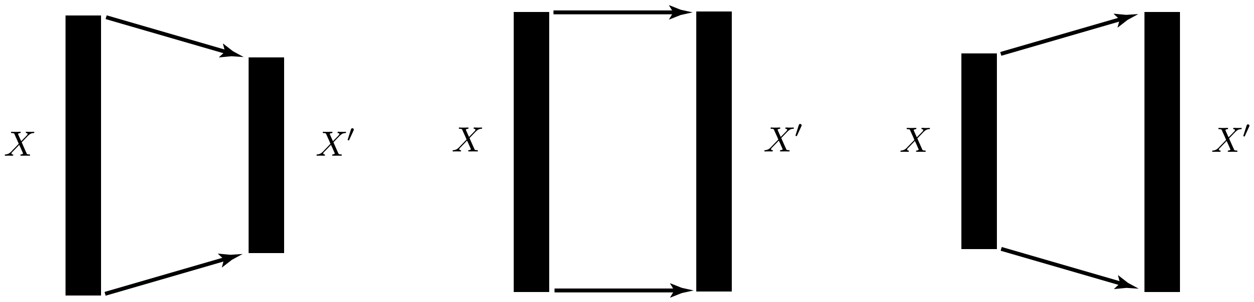
$$H(X) = -\sum_{x} p(x) \log p(x) \qquad H(X') = -\sum_{x'} p(x') \log p(x')$$



$$p(x, x', c) = p(x)p(c)p(x'|x, c)$$

Control dynamics and entropy

Dissipative **Conservative** **Expanding**



Dissipative

- $\Delta H_{\text{open}} > 0$
- Many-to-one mapping
- Damping, friction

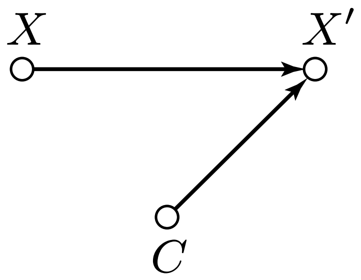
Conservative

- $\Delta H_{\text{open}} = 0$
- One-to-one mapping
- Hamiltonian, closed system

Expanding

- $\Delta H_{\text{open}} < 0$
- One-to-many mapping
- Noise, chaos

Open-loop entropy control



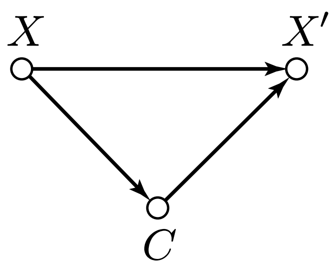
Deterministic: fix c ΔH^c_{open}
Random: fix $p(c)$ ΔH_{open}

Theorem

$$\Delta H_{open} \leq \max_c \Delta H^c_{open}$$

- Follows from concavity of $H(X)$
- Random control cannot out-performed deterministic control
- Random actions increase entropy

Closed-loop or feedback control



$$p(x, x', c) = p(x)p(c|x)p(x'|x, c)$$

Entropy reduction

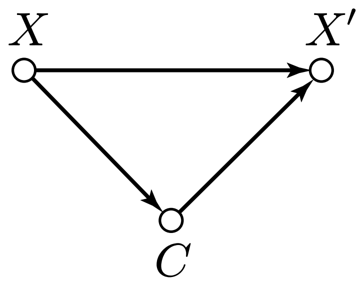
$$\Delta H_{\text{closed}} = H(X) - H(X')_{\text{closed}}$$

Sensor information

$$I(X; C) = \sum_{x,c} p(x, c) \log \frac{p(x, c)}{p(x)p(c)}$$

- Correlation between X and C
- Channel capacity in bits (Shannon)

Closed-loop entropy control



Theorem

$$\Delta H_{\text{closed}} \leq \Delta H_{\text{open}}^{\max} + I(X;C)$$

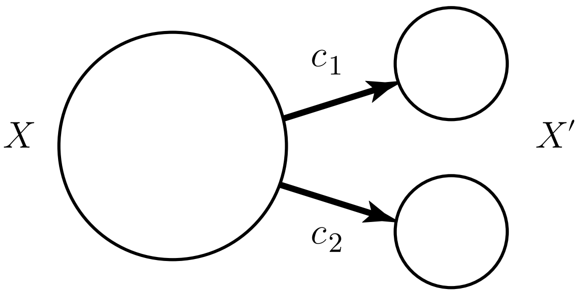
where

$$\Delta H_{\text{open}}^{\max} = \max_{X,c} \Delta H_{\text{open}}$$

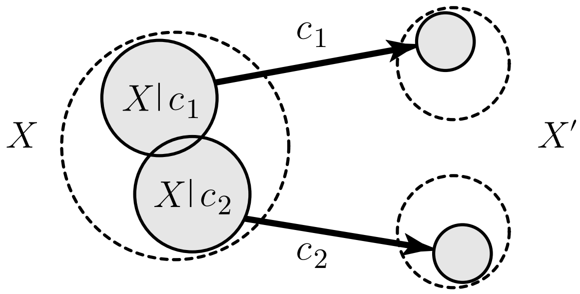
- Limit on entropy reduction
- Equality: optimal controller
- $\Delta H_{\text{closed}} > 0$ if $I(X:C) > -\Delta H_{\text{open}}$
- Can reduce entropy with entropy-increasing actions

Main idea: Conditional analysis

Open-loop

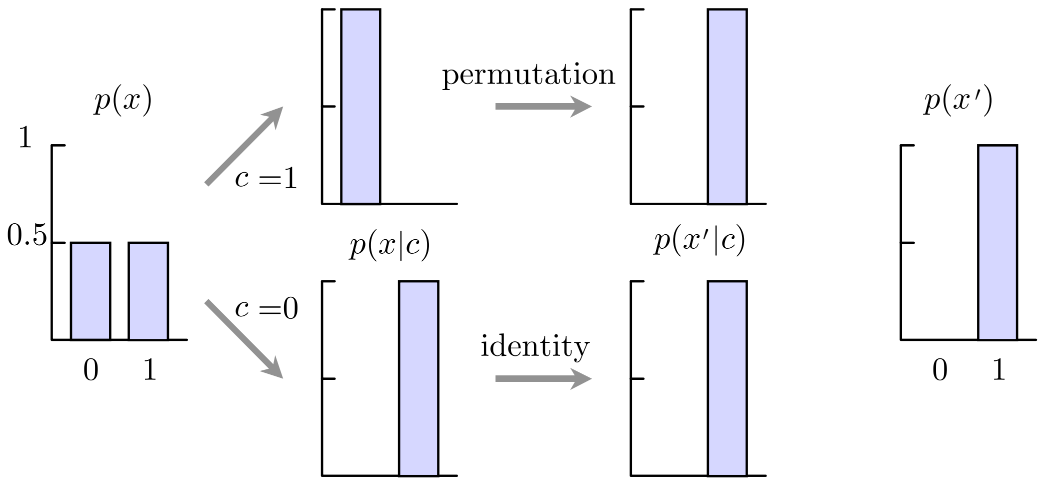


Closed-loop



- Closed-loop control = control based on information
- Observation narrows down the initial state
- Closed-loop acts on smaller sets of initial states
  - Open-loop acts on X
  - Closed-loop acts on $X|c$
  - Entropy difference: $H(X) - H(X|C) = I(X;C)$

Example: Two-state controller



$H(X) = 1$ $\qquad$ $H(X|c) = 0$ $\qquad$ $H(X') = 0$

- $\Delta H_{\text{open}} = 0$
- $\Delta H_{\text{closed}} = I(X;C) = 1$ bit
- Controller is optimal

Applications

- Linear controllers
- Control of chaotic maps (OGY)
  (HT & Lloyd 2004)
- Quantum control
  (Kawabata 2003)
- Adiabatic feedback control
  (Allahverdyan & Saakian 2008)
- Adaptive controllers / robots
  (Polani & Nehaniv, Univ. Hertfordshire)
- Stochastic ratchets / Brownian motors
  (Cao, Feito & HT 2009)
- Cooling systems:

$$\Delta Q_{\text{closed}} \leq \Delta Q_{\text{open}} + \underbrace{k_B T I \ln 2}_{\text{Landauer's cost}}, \qquad \Delta Q = k_B T \Delta H$$

Chaos control

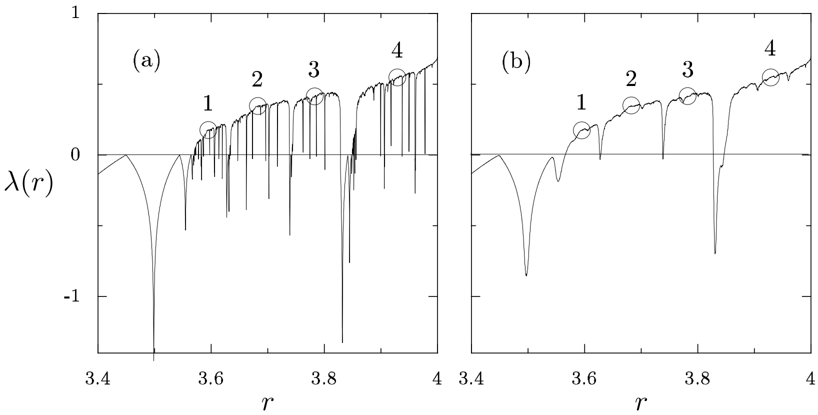
(Ott, Grebogi, Yorke 1990)

- Chaotic map:

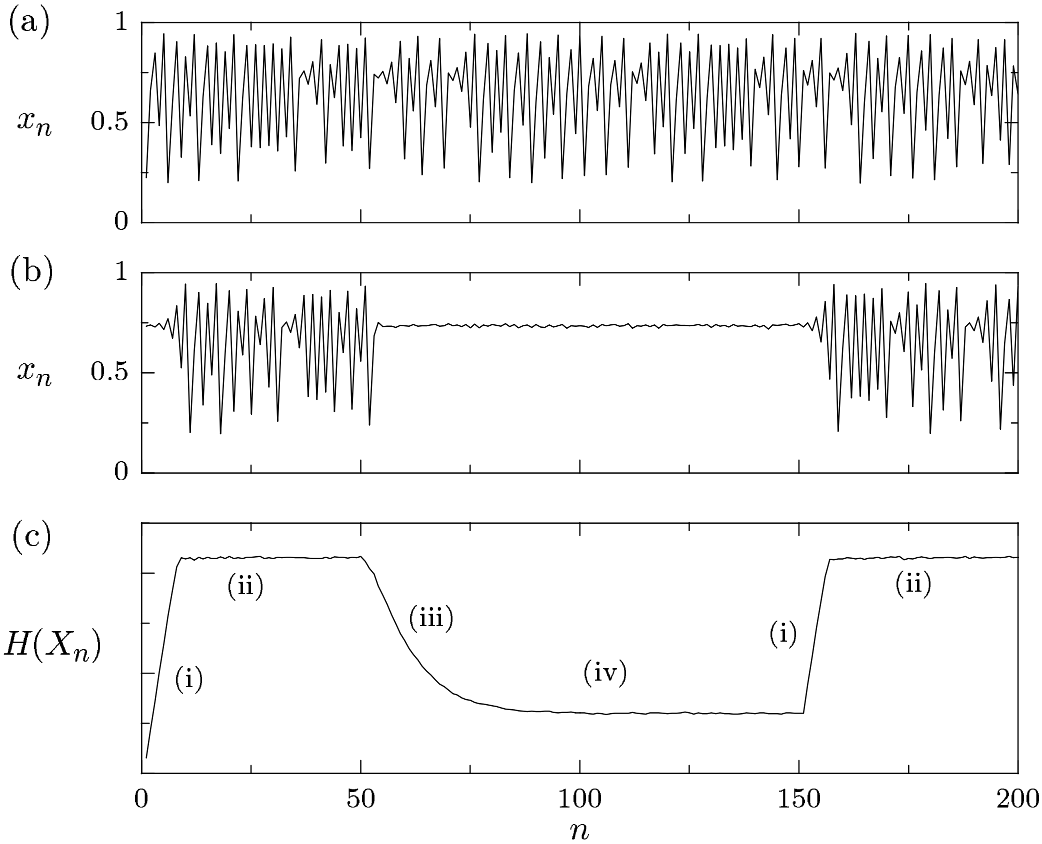
$$x_{n+1} = f(x_n) = r\, x_n(1 - x_n), \qquad x_n \in [0,1], \quad r \in [0,4]$$

- Controlled map (OGY):

$$x_{n+1} = r(x_n)\, x_n(1 - x_n) \qquad \tilde{x}_n : \text{estimate of } x_n$$
$$x^* : \text{target state}$$
$$r(x_n) = r - \gamma(\tilde{x}_n - x^*)$$

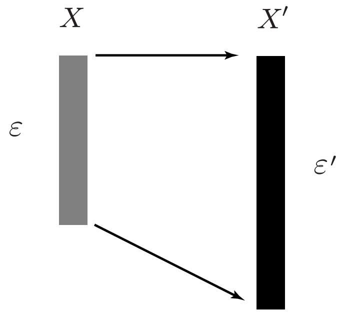


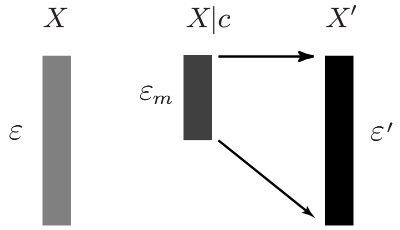
Chaos control (cont'd)

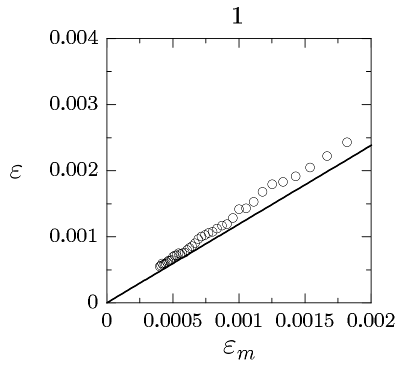


Chaos control (cont'd)

(HT & Lloyd 2000, 2004)







Open-loop

$$\varepsilon' \approx e^{\lambda}\varepsilon$$
$$\lambda > 0$$

$$\Delta H_{\text{open}} = \log\varepsilon - \log\varepsilon'$$
$$= -\lambda$$
$$< 0$$

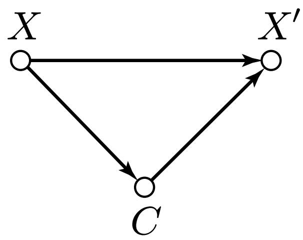
Closed-loop

$$\varepsilon' \approx e^{\lambda}\varepsilon_m$$
$$\Delta H_{\text{closed}} = \log\varepsilon - \log\varepsilon'$$

$$\varepsilon = \varepsilon', \quad \Delta H_{\text{closed}} = 0$$
$$\varepsilon_m = e^{-\lambda}\varepsilon$$
$$I = \log \varepsilon/\varepsilon_m$$

Conclusion

- Control = entropy reduction
- Information in control = $I(X;C)$
- Controller = Maxwell demon
- $\Delta H_{\text{closed}} \leq \Delta H_{\text{open}}^{\max} + I(X;C)$



Future work

- Other systems
  - Many time-steps
  - Memory, non-Markovian correlations
  - Continuous time
  - Quantum systems (with coherent control)
  - Stigmery: use environment to transfer info
  - Sensor-actuator evolution
- Quantities other than ΔH
- Information in stochastic thermodynamics

Future work: More general framework

- Cost functional: $A[x]$
- Compare A_{closed}, A_{open} and I
- Optimal control:

$$A_I = \inf_{\substack{\text{control designs} \\ \text{info} < I}} A[x]$$

- Related to rate distortion theory
- Example:

$\langle e^{-\beta W_T} \rangle_{\text{open}}$ (Jarzynski 1997)
$\langle e^{-\beta W_T} \rangle_{\text{closed}}$ (Sagawa & Ueda 2010)

General approach

- Optimal control theory
- Conditional analysis
  - Open-loop acts on X
  - Closed-loop acts on $X|c$

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